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The space of super light rays for complex conformal spacetimes

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After defining a superconformal structure on a $4|4N$ supermanifold, its space of super light rays is constructed and shown to have a natural supercontact structure. We next construct, for a $5|2N$ dimensional supercontact manifold, its space of *normal quadrics*. This is shown to be a $4|4N$ superconformal manifold. Every four dimensional conformal manifold is then proved to have an extension to a superconformal manifold of dimension $4|4N$ for $N \leq 4$. After the equivalence of the $N = 3$ Supersymmetric Yang–Mills equations and integrability along super light rays is shown, a one to one correspondence between solutions to the $N = 3$ Supersymmetric Yang–Mills equations and certain vector bundles over the $N = 3$ space of super light rays is established.

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Contents

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1. Introduction

1.1. BACKGROUND

In 1978 Witten [22] showed that there is a one to one correspondence between solutions to the $N = 3$ Supersymmetric Yang–Mills (SSYM) equations on complex Minkowski space and certain holomorphic vector bundles over a specific supermanifold, *the space of super light rays*. This result is in the spirit of Roger Penrose's Twistor theory: Analysis on one space is replaced by complex geometry on another, albeit in this case a superspace. The main result of the present paper is an extension of this result to complex conformal spacetimes with general curvature.

The original Ward Correspondence was produced by Richard Ward [20] in 1977, relating instantons on a self-dual complex conformal spacetime to certain vector bundles over its twistor space. (Self-duality refers to a restriction on the curvature of the spacetime, which actually ensures the existence of a twistor space.) This result led to a complete classification of instantons on S^4, since the corresponding vector bundles over its twistor space, $\mathbb{C}P_3$, could be studied using techniques from algebraic geometry.

Shortly after, Isenberg, Yasskin and Green [8], and also independently Witten [22], produced a Generalized Ward Correspondence for the full Yang–Mills equation on Minkowski space. Solutions here correspond to certain vector bundles on the third infinitesimal neighborhood of the space of null geodesics embedded in $\mathbb{C}P_3 \times \mathbb{C}P_3$. Finally, in 1986, LeBrun [12] extended this result to "self-dual" complex conformal spacetimes.

The present work relies in no small part on LeBrun's result [13] that ambitwistor spaces may always be *thickened* up to order 4. If the Bach tensor vanishes, it may be thickened up to order 5, and if the Eastwood–Dighton tensor vanishes, up to order 6. This result was in turn based upon the linear version given by Baston and Mason [1]. A linear version was also given in the supersymmetric setting by Chau and Lim [5]. The role of the Bach tensor as a Yang–Mills current was originally discovered by Merkulov [16].

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1.2. SUPERMANIFOLDS

Let us recall the definitions of superalgebras and supermanifolds. A superalgebra or $\mathbb{Z}_2$-graded commutative algebra is an algebra in which every element can be written as a sum of an *even* element and an *odd* element. Even elements commute with all elements in the algebra and odd elements anticommute with odd elements.

A complex supermanifold is a pair (X, A) where X is a complex manifold and A is a sheaf of $\mathbb{Z}_2$-graded algebras over $\mathbb{C}$ which is locally isomorphic to $\bigwedge_{\mathcal{O}}^{*} \mathcal{O}^{\oplus m}$. We also require that globally $A/N \cong \mathcal{O}$ and that N/N^2 is a locally free sheaf of $\mathcal{O}$-modules. Locally, sections of A on a coordinate neighborhood U will have the form:

$$g = \sum_I g_I \eta^I ,$$

where $g_I = g_I(z^1, z^2, \ldots, z^n) \in \mathcal{O}(U)$ and $\eta^1, \ldots, \eta^m$ are linearly independent sections of $\mathcal{O}^{\oplus m}$. The $z^1, \ldots, z^n$ and $\eta^1, \ldots, \eta^m$ are referred to respectively as the even and odd complex coordinates.The $\mathbb{Z}_2$-grading on A is represented locally by: g is even if

$$g = \sum_{|I| \text{ even}} g_I \eta^I$$

and g is odd if

$$g = \sum_{|I| \text{ odd}} g_I \eta^I .$$

Note that a change of coordinates is required to preserve the $\mathbb{Z}_2$-grading.

One defines super vector bundles as locally free sheaves of A-modules and the supertangent bundle as the sheaf of derivations of superfunctions over $\mathbb{C}$. The supertangent bundle is then a supervector bundle. One may extend many of the ideas of differential geometry, such as differential forms, and the Frobenius theorem, to supergeometry. (We refer the reader to Kostant [10].)

2. Superconformal manifolds

2.1. SUPERCONFORMAL STRUCTURES

A superconformal structure on a $4|4N$ supermanifold is defined by the existence of supervector bundles $S_+^{2|0}, S_-^{2|0}, E^{0|N}$ and the exact sequence

$$0 \to T_l M \oplus T_r M \to TM \to T_0 M \to 0 ,$$

where we have isomorphisms

$$T_l M \cong S_+ \otimes E, \qquad T_r M \cong S_- \otimes E^*, \qquad T_0 M \cong S_+ \otimes S_- .$$

$T_l M$ and $T_r M$ are required to be integrable distributions and the Frobenius form

$$\Phi : T_l M \otimes T_r M \to T_0 M ,$$

where

$$\Phi(X \otimes Y) = [X, Y] \bmod (T_l M \oplus T_r M)$$

is required to coincide via the above isomorphisms with the convolution:

$$S_+ \otimes E \otimes E^* \otimes S_- \to S_+ \otimes S_- .$$

The Frobenius form is then said to be nondegenerate. (These are the only curvature conditions necessary to construct the space of super light rays.)

We refer the reader to Manin [15, pp. 277, 278] for the definition of an $N = 1$ superconformal structure for which the above definition for any N is a simple generalization. The $N = 1$ definition is based on the work of Ogievetskii and Sokachev [18]. Merkulov [17] has also generalized this definition to the case of N-extended paraconformal supermanifolds.

Since $T_l M$ and $T_r M$ are integrable distributions we may define

$$M_l = (M_{rd}, \mathrm{Ker}(T_r M)), \qquad M_r = (M_{rd}, \mathrm{Ker}(T_l M)).$$

We then have the double fibration $M \to M_{l,r}$. The local coordinates $x_l^a, \theta^{\alpha i}$ and $x_r^a, \theta_j^{\dot\alpha}$ on M_l and M_r respectively pullback to functions on M. Define the functions

$$x^a = \frac{x_l^a + x_r^a}{2}, \qquad H^a = \frac{x_l^a - x_r^a}{2\mathrm{i}}.$$

Take $x^a, \theta^{\alpha i}, \theta_j^{\dot\alpha}$ as local coordinates on M. Note that the functions H^a are nilpotent since $(x_l^a)_{rd} = (x_r^a)_{rd} = x_{rd}^a$. Also define the functions

$$X^a{}_{\beta j} = \mathrm{i}\left[\left(I - \mathrm{i}\frac{\partial H}{\partial x}\right)^{-1}\right]^a_c \frac{\partial H^c}{\partial \theta^{\beta j}}, \qquad X^{ja}_{\dot\beta} = -\mathrm{i}\left[\left(I + \mathrm{i}\frac{\partial H}{\partial x}\right)^{-1}\right]^a_c \frac{\partial H^c}{\partial \theta_j^{\dot\beta}}.$$

The derivations

$$q_{\alpha j} = \frac{\partial}{\partial \theta^{\alpha j}} + X^b_{\alpha j}\frac{\partial}{\partial x^b}, \qquad q^j_{\dot\alpha} = \frac{\partial}{\partial \theta^{\dot\alpha}_j} + X^{jb}_{\dot\alpha}\frac{\partial}{\partial x^b},$$

then form a local basis for $T_l M$ and $T_r M$ respectively and the one-forms

$$\omega^a = dx^a - d\theta^{\beta j} X^a_{\beta j} - d\theta_j^{\dot\beta} X^{ja}_{\dot\beta}$$

form a local basis for $\Omega_0^1 M$. (See Manin [15, p. 281].)

The space of super light vectors, Σ, is defined as a submanifold of $\Omega_0^{1'} M$:

$$\Sigma = \{v \in \Omega_0^{1'} M \mid v = s_+ \otimes s_-, s_+ \in S_+^*, s_- \in S_-^*\}.$$

(Here, $'$ denotes removal of the zero section.) Let $(x^a, \theta^{\alpha j}, \theta_j^{\dot\alpha}, \xi_a)$ be local coordinates on $\Omega_0^{1'} M$ where a local section of $\Omega_0^{1'} M$ over M is given by $\xi_a \omega^a$. On $\Omega_0^{1'} M$, $\xi_a \omega^a$ is a canonical one-form. $d(\xi_a \omega^a)$ is called the standard *pre-symplectic* form on $\Omega_0^{1'} M$. We proceed with a *pre-symplectic reduction* on $\Omega_0^{1'} M$ to construct our space of super light rays for $M^{4|4N}$.

2.2. THE KERNEL OF THE PRE-SYMPLECTIC FORM

Denote the isomorphism $T_l M \cong S_+ \otimes E$ by

$$s_\alpha^+ \otimes e_j = g_{\alpha j}^{\beta k} q_{\beta k} ,$$

the isomorphism $T_r M \cong S_- \otimes E^*$ by

$$s_{\dot\alpha}^- \otimes e^j = f_{\dot\alpha k}^{j\dot\beta} q_{\dot\beta}^k$$

and the isomorphism $T_0 M \cong S_+ \otimes S_-$ by

$$s_\alpha^+ \otimes s_{\dot\alpha}^- = h_{\alpha\dot\alpha}^b \,\partial/\partial x^b .$$

The condition that the Frobenius form coincides with convolution via these isomorphisms is that:

$$[g_{\alpha j}^{\beta i} q_{\beta i}, f_{\dot\alpha k}^{l\dot\beta} q_{\dot\beta}^k] = h_{\alpha\dot\alpha}^c \frac{\partial}{\partial x^c}\delta_j^l \bmod (T_l M \oplus T_r M) ,$$

i.e.,

$$g_{\alpha j}^{\beta i} f_{\dot\alpha k}^{l\dot\beta} \Phi_{\beta\dot\beta i}^{ck} h_c^{-1\sigma\dot\sigma} = \delta_\alpha^\sigma \delta_{\dot\alpha}^{\dot\sigma} \delta_j^l ,$$

where

$$[q_{\beta i}, q_{\dot\beta}^k] = \Phi_{\beta\dot\beta i}^{ck} \frac{\partial}{\partial x^c} \bmod (T_l M \oplus T_r M) .$$

It is a straightforward calculation, using the definitions of $q_{\alpha i}$ and $X_{\alpha i}^c$, to show that

$$[q_{\alpha i}, q_{\beta k}] = (q_{\beta j} X_{\alpha i}^c + q_{\beta j} X_{\beta i}^c) \frac{\partial}{\partial x^c} = 0 .$$

Similarly,we have

$$[q_{\dot\alpha}^i, q_{\dot\beta}^j] = (q_{\dot\alpha}^i X_{\dot\beta}^{cj} + q_{\dot\beta}^j X_{\dot\alpha}^{ci}) \frac{\partial}{\partial x^c} = 0 .$$

The pre-symplectic form $d\xi_a \wedge \omega^a + \xi_a \wedge d\omega^a$ is then

$$\begin{aligned}
& d\xi_a \wedge \omega^a + \xi_a \wedge (d\theta^{\beta j} \wedge dX_{\beta j}^a + d\theta_j^{\dot\beta} \wedge dX_{\dot\beta}^{aj}) \\
&\quad = d\xi_a \wedge \omega^a - \xi_a d\theta^{\beta j} \wedge \omega^c \partial X_{\beta j}^a/\partial x^c - \xi_a d\theta_j^{\dot\beta} \wedge \omega^c \partial X_{\dot\beta}^{aj}/\partial x^c \\
&\qquad - \xi_a d\theta^{\beta j} \wedge d\theta_k^{\dot\beta} (q_{\beta j} X_{\dot\beta}^{ak} + q_{\dot\beta}^k X_{\beta j}^a) \\
&\qquad - \xi_a d\theta^{\beta j} \wedge d\theta^{\gamma k} (q_{\beta j} X_{\gamma k}^a + q_{\gamma k} X_{\beta j}^a) - \xi_a d\theta_j^{\dot\beta} \wedge d\theta_k^{\dot\gamma} (q_{\dot\beta}^j X_{\dot\gamma}^{ak} + q_{\dot\gamma}^k X_{\dot\beta}^{aj}) .
\end{aligned}$$

The last two terms are zero and the fourth is $-d\theta^{\beta j} \wedge d\theta_k^{\dot\beta} \Phi_{\beta\dot\beta j}^{ak} \xi_a$. So we have altogether for the pre-symplectic form:

$$\begin{aligned}
d(\xi_a \omega^a) = {} & d\xi_a \wedge \omega^a - d\theta^{\beta j} \wedge \omega^a \xi_c \frac{\partial X_{\beta j}^c}{\partial x^a} \\
& - d\theta_j^{\dot\beta} \wedge \omega^a \xi_c \frac{\partial X_{\dot\beta}^{cj}}{\partial x^a} - d\theta^{\beta j} \wedge d\theta_k^{\dot\beta} \Phi_{\beta\dot\beta j}^{ak} \xi_a .
\end{aligned}$$

Pull this form back to

$$\Sigma = \{h^a_{\alpha\dot\alpha}\xi_a h^b_{\beta\dot\beta}\xi_b \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} = 0\}$$

(where $\epsilon^{00} = \epsilon^{11} = 0$ and $\epsilon^{01} = -\epsilon^{10} = 1$ and similarly for $\epsilon^{\dot\alpha\dot\beta}$), so as to find the kernel of the pre-symplectic form restricted to Σ.

Let $D_{\sigma l} = g^{\beta k}_{\sigma l} q_{\beta k}$ and $D^l_{\dot\sigma} = f^{l\dot\beta}_{\dot\sigma j} q^j_{\dot\beta}$. Also let $\zeta_{\mu\dot\mu} = h^c_{\mu\dot\mu}\xi_c$. We claim that the vector fields

$$Q_l = \epsilon^{\sigma\mu}\zeta_{\mu\dot\mu}\left(D_{\sigma l} - g^{\beta k}_{\sigma l}\xi_c \frac{\partial X^c_{\beta k}}{\partial x^a}\frac{\partial}{\partial \xi_a}\right),$$

$$Q^l = \epsilon^{\dot\sigma\dot\mu}\zeta_{\mu\dot\mu}\left(D^l_{\dot\sigma} - f^{l\dot\beta}_{\dot\sigma j}\xi_c \frac{\partial X^{cj}_{\dot\beta}}{\partial x^a}\frac{\partial}{\partial \xi_a}\right)$$

are in the kernel of $d(\xi_a\omega^a)$. This calculation will be left to the reader.

We must show that Q_l and Q^l are tangent to the space of super light vectors, Σ, i.e. that

$$Q_l(\zeta_{\nu\dot\nu}\zeta_{\mu\dot\mu}\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu}) = Q^l(\zeta_{\nu\dot\nu}\zeta_{\mu\dot\mu}\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu}) = 0$$

on Σ:

$$\begin{aligned}
&Q_l(\zeta_{\nu\dot\nu}\zeta_{\mu\dot\mu}\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu}) \\
&= \epsilon^{\sigma\kappa}\zeta_{\kappa\dot\kappa}g^{\beta k}_{\sigma l}\left(q_{\beta k} - \xi_c\frac{\partial X^c_{\beta k}}{\partial x^a}\frac{\partial}{\partial\xi_a}\right)(h^d_{\nu\dot\nu}\xi_d h^b_{\mu\dot\mu}\xi_b\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu}) \\
&= \epsilon^{\sigma\kappa}\zeta_{\kappa\dot\kappa}g^{\beta k}_{\sigma l}\left((q_{\beta k}h^d_{\nu\dot\nu})\xi_d\zeta_{\mu\dot\mu} - \xi_c\frac{\partial X^c_{\beta k}}{\partial x^a}h^a_{\nu\dot\nu}\zeta_{\mu\dot\mu}\right)\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu} \\
&\quad + \epsilon^{\sigma\kappa}\zeta_{\kappa\dot\kappa}g^{\beta k}_{\sigma l}\left((q_{\beta k}h^d_{\mu\dot\mu})\xi_d\zeta_{\nu\dot\nu} - \xi_c\frac{\partial X^c_{\beta k}}{\partial x^a}h^a_{\mu\dot\mu}\zeta_{\nu\dot\nu}\right)\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu} \\
&= 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot\kappa}\left(g^{\beta l}_{\sigma l}q_{\beta k}h^d_{\nu\dot\nu} - g^{\beta k}_{\sigma l}\frac{\partial X^d_{\beta k}}{\partial x^a}h^a_{\nu\dot\nu}\right)\zeta_{\mu\dot\mu}\xi_d\epsilon^{\mu\nu}\epsilon^{\dot\mu\dot\nu}.
\end{aligned}$$

To show this is zero, consider the quantity

$$R^d_{\sigma l\nu\dot\nu} = g^{\beta k}_{\sigma l}q_{\beta k}h^d_{\nu\dot\nu} - g^{\beta k}_{\sigma l}\frac{\partial X^d_{\beta k}}{\partial x^a}h^a_{\nu\dot\nu} + g^{\beta k}_{\sigma l}(q_{\beta k}g^{\mu m}_{\nu l})g^{-1\alpha l}_{\mu m}h^d_{\alpha\dot\nu}.$$

We claim that $R^d_{\sigma l\nu\dot\nu} = -R^d_{\nu l\sigma\dot\nu}$ and thus $R^d_{\sigma l\nu\dot\nu} = \mathcal{R}^d_{l\dot\nu}\epsilon_{\sigma\nu}$ This follows from the Bianchi identity:

$$\begin{aligned}
0 = &[g^{\beta k}_{\sigma l}q_{\beta k}, [g^{\gamma m}_{\nu l}q_{\gamma m}, f^{l\dot\rho}_{\dot\nu n}q^n_{\dot\rho}]] \\
&+ [f^{l\dot\rho}_{\dot\nu n}q^n_{\dot\rho}, [g^{\beta k}_{\sigma l}q_{\beta k}, g^{\gamma m}_{\nu l}q_{\gamma m}]] + [g^{\gamma m}_{\nu l}q_{\gamma m}, [f^{l\dot\rho}_{\dot\nu n}q^n_{\dot\rho}, g^{\beta k}_{\sigma l}q_{\beta k}]]
\end{aligned}$$

(we do not sum over l)

$$= [g^{\beta k}_{\sigma l} q_{\beta k}, h^{d}_{\nu\dot{\nu}} \frac{\partial}{\partial x^{d}}] + [f^{l\dot{\rho}}_{\dot{\nu}n} q^{n}_{\dot{\rho}}, g^{\beta k}_{\sigma l} (q_{\beta k} g^{\gamma m}_{\nu l}) q_{\gamma m})] + (\sigma \leftrightarrow \nu)$$

$$= (g^{\beta k}_{\sigma l} q_{\beta k} h^{d}_{\nu\dot{\nu}} - g^{\beta k}_{\sigma l} h^{a}_{\nu\dot{\nu}} \frac{\partial X^{d}_{\beta k}}{\partial x^{a}} + g^{\beta k}_{\sigma l} (q_{\beta k} g^{\gamma m}_{\nu l}) f^{l\dot{\mu}}_{\dot{\nu}n} \Phi^{dn}_{\gamma\dot{\mu}m}) \frac{\partial}{\partial x^{d}}$$

$$+ (\sigma \leftrightarrow \nu) \bmod (T_l M \oplus T_r M) .$$

But $f^{l\dot{\mu}}_{\dot{\nu}n} \Phi^{dn}_{\gamma\dot{\mu}m} = g^{-1\alpha l}_{\gamma m} h^{d}_{\alpha\dot{\nu}}$, so we obtain

$$0 = (R^{d}_{\sigma l\nu\dot{\nu}} + R^{d}_{\nu l\sigma\dot{\nu}}) \partial/\partial x^{d} .$$

Thus $R^{d}_{\sigma l\nu\dot{\nu}} = -R^{d}_{\nu l\sigma\dot{\nu}}$.

Now

$$\begin{aligned}
&Q_l(\zeta_{\nu\dot{\nu}}\zeta_{\mu\dot{\mu}}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}})\\
&\quad = 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot{\kappa}}R^{b}_{\sigma l\nu\dot{\nu}}\zeta_{\mu\dot{\mu}}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}}\xi_b\\
&\qquad - 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot{\kappa}}g^{\beta k}_{\sigma l}(q_{\beta k}g^{\gamma m}_{\nu l})g^{-1\alpha l}_{\gamma m}h^{b}_{\alpha\dot{\nu}}\zeta_{\mu\dot{\mu}}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}}\xi_b\\
&\quad = 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot{\kappa}}\mathcal{R}^{b}_{l\dot{\nu}}\epsilon_{\sigma\nu}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}}\zeta_{\mu\dot{\mu}}\xi_b\\
&\qquad - 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot{\kappa}}g^{\beta k}_{\sigma l}(q_{\beta k}g^{\gamma m}_{\nu l})g^{-1\alpha l}_{\gamma m}\zeta_{\alpha\dot{\nu}}\zeta_{\mu\dot{\mu}}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}}\\
&\quad = 2\epsilon^{\mu\kappa}\epsilon^{\dot{\mu}\dot{\nu}}\mathcal{R}^{b}_{l\dot{\nu}}\eta_\kappa \upsilon_{\dot{\kappa}}\eta_\mu \upsilon_{\dot{\mu}}\xi_b\\
&\qquad - 2\epsilon^{\sigma\kappa}\zeta_{\kappa\dot{\kappa}}g^{\beta k}_{\sigma l}(q_{\beta k}g^{\gamma m}_{\nu l})g^{-1\alpha l}_{\gamma m}\eta_\alpha \upsilon_{\dot{\nu}}\eta_\mu \upsilon_{\dot{\mu}}\epsilon^{\mu\nu}\epsilon^{\dot{\mu}\dot{\nu}} = 0,
\end{aligned}$$

since $\epsilon^{\mu\kappa}\eta_\kappa\eta_\mu = \epsilon^{\dot{\mu}\dot{\nu}}\upsilon_{\dot{\nu}}\upsilon_{\dot{\mu}} = 0$ and where we have written $\zeta_{\kappa\dot{\kappa}} = \eta_\kappa \upsilon_{\dot{\kappa}}$ for some spinor fields η_κ and $\upsilon_{\dot{\kappa}}$.

There is a similiar result for Q^l and hence $[Q_l, Q^l]$ is also tangent to Σ. In addition, $[Q_l, Q^l]$ is in the kernel of $d(\xi_a\omega^a)|_\Sigma$ since the kernel of a closed two-form is closed under Lie brackets. We claim that the set $\{Q^l, Q_l, P = \sum_k [Q_k, Q^k]\}$ forms a basis for $\ker(d(\xi_a\omega^a)|_\Sigma)$ and thus that this kernel has rank $1|2N$.

Now

$$\operatorname{rank}(\ker(d(\xi_a\omega^a)|_\Sigma)) \leq \operatorname{rank}(\ker(\{d(\xi_a\omega^a)|_\Sigma\}_{rd})),$$

where $\{d(\xi_a\omega^a)\}_{rd}$ is the *reduction* composed with $d(\xi_a\omega^a)|_\Sigma$ and takes sections of $T\Sigma$ to sections of $(\Omega^1\Sigma)_{rd}$.

We have

$$\begin{aligned}
\{d(\xi_a\omega^a)\}_{rd} &= d\xi_a \wedge dx^a + d\theta^{\beta j} \wedge \theta^{\dot{\beta}}_k \Phi^{ak}_{\beta\dot{\beta}j}\xi_a\\
&= d\xi_a \wedge dx^a + D^{\alpha k} \wedge D^{\dot{\alpha}}_k \eta_\alpha \upsilon_{\dot{\alpha}},
\end{aligned}$$

where $D^{\alpha k}$ and $D^{\dot{\alpha}}_k$ are dual to $D_{\alpha k}$ and $D^k_{\dot{\alpha}}$. Now

$$(T\Sigma)_{rd} = (T_0\Sigma)_{rd} \oplus (\pi^* T_l M \oplus \pi^* T_r M)_{rd},$$

where π is the natural projection to M and our bilinear form is actually a direct sum of two bilinear forms, the two terms just written above. The first term has kernel of rank 1, while the second has kernel of rank $2N$. Thus $\mathrm{rank}(\ker(d(\xi_a\omega^a)|_\Sigma)) \leq 1|2N$. Since

$$([Q^l, Q_l])_{rd} = (h^a_{\beta\dot\beta})_{rd}\frac{\partial}{\partial x^a} + A_a\frac{\partial}{\partial \xi_a} \neq 0$$

for some quantity A_a, we have that this rank is actually equal to $1|2N$. $\mathrm{Ker}(d(\xi_a\omega^a)|_\Sigma)$ is then a distribution. It is also an integrable distribution since, as stated before, the kernel of a closed two-form is closed under Lie brackets. The space of super light rays will be constructed from the leaf space of this distribution. It will therefore be useful to inquire into this in the following section.

2.3. A LEMMA ON LEAF SPACES FOR SUPERMANIFOLDS

Let $\mathcal{D}^{n-p|m-q}$ be an integrable distribution on a complex supermanifold, $Y^{n|m}$. The reduction of $\mathcal{D}$ splits into an even and an odd part:

$$\mathcal{D}_{rd} = \mathcal{D}_{rd\,0} \oplus \mathcal{D}_{rd\,1}\,.$$

$\mathcal{D}_{rd\,0}$ is an integrable distribution on Y_{rd}. Assume that the leaf space of $\mathcal{D}_{rd\,0}$ is a complex manifold, X^p_{rd}, and thus that we have a holomorphic map $\rho_{rd} : Y_{rd} \to X_{rd}$ whose fibres are the leaves of $\mathcal{D}_{rd\,0}$. We wish to examine some sufficient conditions under which ρ_{rd} extends to a map, ρ, onto some complex supermanifold, X, such that the fibres of ρ are the leaves of $\mathcal{D}$.

Let $\mathcal{B} = \rho_{rd*}(\ker\mathcal{D})$, the push down of the sheaf of superfunctions on Y, which are annihilated by $\mathcal{D}$. Of course, $\rho^{-1}_{rd}\mathcal{B} = \ker\mathcal{D}$. We will show that under appropriate conditions, $X = (X_{rd}, \mathcal{B})$ is the complex supermanifold that we seek and thus that the canonical identification $\rho^{-1}_{rd}\mathcal{B} = \ker\mathcal{D}$ defines our map ρ between supermanifolds.

We need to show that $\mathcal{B}$ is isomorphic locally to $\bigwedge^\bullet \mathcal{O}^{\oplus q}_{X_{rd}}$. It is sufficient therefore to assume X_{rd} is a contractable Stein domain and thus that Y has a covering by Frobenius charts, $\{U_\alpha\}$, such that the even coordinates satisfy $x_{\alpha\,rd} = x_{\beta\,rd}$. We wish to show on this Y that $\rho^{-1}_{rd}\mathcal{B}$ is globally isomorphic to $\bigwedge^\bullet \rho^{-1}_{rd}\mathcal{O}^{\oplus q}_{X_{rd}}$. This will give the local splitting of $\mathcal{B}$ on X.

We first observe that on Y, $\rho^{-1}_{rd}\mathcal{B}$ and $\bigwedge^\bullet \rho^{-1}_{rd}\mathcal{O}^{\oplus q}_{X_{rd}}$ are already locally isomorphic. Indeed, this is clear if one restricts oneself to a Frobenius chart where we have local coordinates $x^a, \theta^j, y^b, \phi^k$ such that $\mathcal{D}$ is spanned by $\partial/\partial y^b$ and $\partial/\partial\phi^k$. Any change of coordinates on an overlap of two Frobenius charts is an automorphism of $\bigwedge^\bullet \rho^{-1}_{rd}\mathcal{O}^{\oplus q}_{X_{rd}}$, as a $\mathbb{Z}_2$-graded algebra, which leaves fixed $\rho^{-1}_{rd}\mathcal{O}_{X_{rd}} \subset \bigwedge^\bullet \rho^{-1}_{rd}\mathcal{O}^{\oplus q}_{X_{rd}}$. (Note that we have a covering such that $x^a_{\alpha\,rd} = x^a_{\beta\,rd}$.) Let $\mathcal{A}$ denote the sheaf of all such automorphisms. $\rho^{-1}_{rd}\mathcal{B}$ is then given by an element, τ, of the point set $H^1(Y_{rd}, \mathcal{A})$. We wish to examine the structure of $\rho^{-1}_{rd}\mathcal{B}$ order by order. Let Nil denote here the subsheaf of nilpotents of $\bigwedge^\bullet \rho^{-1}_{rd}\mathcal{O}^{\oplus q}_{X_{rd}}$.

Let $\mathcal{A}^{(j)}$ denote the sheaf of automorphisms of $(\bigwedge^{\bullet}\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q})/(\mathrm{Nil})^{j+1}$ which preserve $\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}$. We have $\mathcal{A} = \mathcal{A}^{(q)}$ and a natural map $\mathcal{A}^{(l)} \to \mathcal{A}^{(j)}$ for $l > j$. We have the exact sequences for $j \geq 1$,

$$0 \to \mathcal{C}^{(j)} \to \mathcal{A}^{(j)} \to \mathcal{A}^{(j-1)} \to 0 .$$

The structures of $\mathcal{C}^{(j)}$ have been given by Batchelor [2]. (See also Eastwood and LeBrun [6].) They are

$$\begin{aligned}
\mathcal{C}^{(1)} &= \mathcal{A}^{(1)} = GL(q, \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}) , \\
\mathcal{C}^{(j)} &= \mathrm{Der}(\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}) \otimes \bigwedge^{j} \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q} \quad \text{for } j \text{ even}, \\
\mathcal{C}^{(j)} &= \mathrm{Hom}(\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}, \bigwedge^{j} \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}) \quad \text{for } j \text{ odd}, \neq 1.
\end{aligned}$$

Let $\tau_{(1)}$ be the image of τ under the natural map

$$H^1(Y_{rd}, \mathcal{A}) \to H^1(Y_{rd}, \mathcal{A}^{(1)}) = H^1(Y_{rd}, GL(q, \rho_{rd}^{-1}\mathcal{O}_{X_{rd}})) .$$

This represents a vector bundle on Y_{rd} and one can check that it is also given by $((TY)_{rd\,1}/\mathcal{D}_{rd\,1})^{*}$. We assume for now that this is a trivial bundle, i.e. $\tau_{(1)} = 1$.

We now apply the machinery in Eastwood and LeBrun [6] of non-abelian sheaf cohomology to each of the exact sequences written before so as to examine the structure of $\rho_{rd}^{-1}\mathcal{B}$ order by order. Assuming inductively that the preceding order gave a trivial structure, i.e. $\tau_{(j-1)} = 1$, This structure is given by

$$\begin{aligned}
&H^1(Y_{rd}, \mathrm{Der}(\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}) \otimes \bigwedge^{j} \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}) \quad \text{for } j \text{ even}, \\
&H^1(Y_{rd}, \mathrm{Hom}(\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}, \bigwedge^{j} \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}) \quad \text{for } j \text{ odd}, \neq 1 .
\end{aligned}$$

These sheaves are inverse images of vector bundles over X_{rd}. By a theorem of Buchdahl [3], these groups are zero if we assume $H^1(\rho_{rd}^{-1}(x), \mathbb{C}) = 0$ for all $x \in X_{rd}$.

We had assumed before that the vector bundle coming from the first order structure was trivial. Let us justify this. One can check that the vector bundle, E, on Y_{rd} given by $\tau_{(1)}$ when restricted to leaves is equipped with a flat connection. This is because sections of $\rho_{rd}^{-1}\mathcal{O}_{X_{rd}}$ are constant on leaves. Assuming that the leaves are simply connected eliminates any holonomy and thus E is the trivial bundle when restricted to a leaf.

Restrict now to a trivializing Frobenius chart, U_α, on Y_{rd} where $E \cong U_\alpha \times \mathbb{C}^q$. Let (x^a, y^b, u^j) be local coordinates for $E|_{U_\alpha}$ with $(x^a = 0, y^b = 0) \in U_\alpha$. Let $W \subset X_{rd}$ be a neighborhood such that $0 \in W$ and $W \subset \rho_{rd}(U_\alpha)$. For all $x^a \in W$ and $u_0 \in \mathbb{C}^q$ there is a unique global section, $V(x^a, y^b)$, of $E|_{\rho_{rd}^{-1}(x^a)}$, with $V_\alpha(x^a, 0) = u_0$. This global section is actually constant when restricted to a leaf. Thus $V_\alpha(x^a, y^b) = u_0$ and we see that $V(x^a, y^b)$ is analytic in x^a. (Recall that all of our transition functions for E are analytic in x^a.) Hence V is a global holomorphic section of $E|_{\rho_{rd}^{-1}(W)}$. Choosing q linearly independent u_0^j will give q linearly independent holomorphic sections V^j of $E|_{\rho_{rd}^{-1}(W)}$, which we conclude is trivial.

We thus obtain that $\rho_{rd}^{-1}\mathcal{B}$ is globally isomorphic to $\bigwedge^\bullet \rho_{rd}^{-1}\mathcal{O}_{X_{rd}}^{\oplus q}$ over W.

Lemma 2.1. *Let $\mathcal{D}$ be an integrable distribution on a complex supermanifold, Y. Assume that the leaves of $\mathcal{D}_{rd\,0}$ are simply connected and that the leaf space of $\mathcal{D}_{rd\,0}$ is a complex manifold, X_{rd}. The leaf space of $\mathcal{D}$ is then a complex supermanifold, X.*

2.4. THE SPACE OF SUPER LIGHT RAYS

We now proceed, almost verbatim, along the lines of LeBrun [11], to define the space of super light rays and to show it has a natural contact structure. Let $\phi = d(\xi_a\omega^a)|_\Sigma$. We suppose that the foliation of the distribution $\ker(\phi)$ satisfies the conditions necessary for its leaf space to be a complex supermanifold. (We, for example, can assume that the null geodesics of the reduced conformal spacetime are simply connected and thus apply lemma 2.1.) Let $\rho : \Sigma \to F$ denote projection; then there is a two-form $\hat{\phi} \in \Gamma(\Omega^2 F)$ such that $\rho^*\hat{\phi} = \phi$. This is true since for $v \in \ker(\phi)$, $L_v\phi = v \sqcup d\phi + d(v \sqcup \phi) = 0$. (Here, $\sqcup$ denotes contraction.) Also $d\hat{\phi} = 0$ since $\rho^* d\hat{\phi} = d\phi = 0$ and ρ being a projection, $\rho^* : \Omega^1 F \to \Omega^1 \Sigma$ is injective. Note that since $\operatorname{rank}(\sqcup\hat{\phi}) = \operatorname{rank}(\sqcup\phi)$, $\sqcup\hat{\phi} : TF \to T^*F$ is an isomorphism ($\det \hat{\phi}_{jk} \neq 0$, $j,k = 1,\ldots,6+2N$).

There is a $\mathbb{C}_*$-action on $\Omega_0^1 M$ given by scalar multiplication ($\mathbb{C} \subset \mathcal{A}$ = sheaf of superfunctions),

$$m_t : (x^a, \theta^{\alpha j}, \theta_j^{\dot\alpha}, \xi_a) \mapsto (x^a, \theta^{\alpha j}, \theta_j^{\dot\alpha}, t\xi_a)\,.$$

We have $m_t^*\phi = t\phi$, so for $v \in \ker\phi$,

$$\phi \sqcup m_{t*}v = m_t^*\phi \sqcup v = t\phi \sqcup v = 0\,.$$

m_{t*} is clearly injective, so $m_{t*}\ker\phi = \ker\phi$ and leaves are taken unto leaves by m_t.

We can then define $\mathcal{N} = F/\mathbb{C}_*$ to be our space of super light rays. Define $L = F \times \mathbb{C}/\mathbb{C}_*$; we have then $F = L^* - \{\text{zero section}\}$. F has a standard $\mathbb{C}_*$-invariant vector field X along the fibers. We define our contact form $\theta \in \Gamma(\Omega^1(L))$ by $\lambda^*\theta = X \sqcup \hat{\phi}$ where $\lambda : F \to \mathcal{N}$ and $m_t^*(X \sqcup \hat{\phi}) = tX \sqcup \hat{\phi}$. We have for σ a local section of $F \to \mathcal{N}$ an identification of θ with $\sigma^*(X \sqcup \hat{\phi})$,

$$\begin{aligned}\sigma^*(X \sqcup \hat{\phi}) \wedge (d(\sigma^*(X \sqcup \hat{\phi})^{\wedge 2+N})) &= \sigma^*(X \sqcup \hat{\phi} \wedge d(X \sqcup \hat{\phi})^{\wedge 2+N}) \\ &= \sigma^*(X \sqcup \hat{\phi} \wedge (L_X\hat{\phi})^{\wedge 2+N}) = \sigma^*(X \sqcup (\hat{\phi})^{\wedge 3+N})\,,\end{aligned}$$

since

$$L_X\hat{\phi} = \frac{d}{dt}(\mathrm{e}^t\hat{\phi})|_{t=0} = \hat{\phi}.$$

But

$$\hat{\phi}^{\wedge 3+N}(X_1, \ldots, X_{6+N}) \neq 0$$

for any local basis $X_1, \ldots, X_{6+N}$ of TF, so $X \sqcup \hat{\phi}^{\wedge 3+N} \neq 0$ and X is transverse to the image of σ so that $\sigma^*(X \sqcup \hat{\phi}^{\wedge 3+N}) \neq 0$. Thus $\theta \wedge (d\theta)^{\wedge 2+N} \neq 0$ and θ is a contact one-form on $\mathcal{N}$.

3. The space of normal quadrics

3.1. SOME RIGIDITY LEMMAS

In this section we present several lemmas that will be very useful when we start to deform normal quadrics. These lemmas are well known in the literature on deformation theory. See, for example, Burns [4, p. 138].

Lemma 3.1. *Let $\mathcal{E} \to X \times U$ be a vector bundle, where U is an open polydisk in $\mathbb{C}^n$ and X is a compact complex manifold. Let $E_u = \mathcal{E}|_{X \times \{u\}}$ and assume $H^1(X, E_{u_0} \otimes E^*_{u_0}) = 0$ for a given u_0. There then exists a neighborhood of u_0, U', such that $E_u \cong E_{u_0}$ for all $u \in U'$. In other words, $\mathcal{E}|_{U'} \cong \mathrm{pr}^* E_{u_0}$ where* pr *is the projection $X \times U \xrightarrow{\mathrm{pr}} X$.*

The above theorem may also be shown in a particular supersymmetric case.

Lemma 3.2. *Let $\mathcal{E} \xrightarrow{p} X \times U$ be a super vector bundle over a supermanifold. We assume here that U is a super polydisk in $\mathbb{C}^{m|q}$, i.e. $U = (U_{rd}, \bigwedge^\bullet \mathcal{O}^{\oplus q}_{U_{rd}})$, where U^m_{rd} is a polydisk in $\mathbb{C}^m$. We also assume that X is purely even (no odd coordinates) and that $X \times U = (X \times U_{rd}, \mathcal{A} = \bigwedge^\bullet \mathcal{O}^{\oplus q}_{X \times U_{rd}})$. Let $E_{u_0} = \mathcal{E}|_{X \times u_0}$ and assume $H^1(X, E_{u_0} \otimes E^*_{u_0}) = 0$. There then exists a (super)neighborhood of u_0, $U' \subset U$, such that $\mathcal{E}|_{U'} \cong \mathrm{pr}^* E_{u_0}$ where* pr *is the projection $X \times U \xrightarrow{\mathrm{pr}} X$.*

To prove this, first apply lemma 3.1 to $\mathcal{E} \to X \times U_{rd}$. Since $\mathcal{E} = \mathrm{pr}^* E_{u_0}$ for U small enough, and $H^1(X, E_{u_0} \otimes E^*_{u_0}) = 0$ we have $H^1(\mathcal{E}_{rd} \otimes \mathcal{E}^*_{rd}) = 0$. Now consider the machinery of Griffiths obstructions given by Eastwood and LeBrun [2]. These are the obstructions to extending $\mathcal{E}|_{X \times U_{rd}}$ to all of $X \times U$. Also, if such an extension exists, one may also measure its possible uniqueness. It is the second question which we are of course interested in. The machinery proceeds as follows.

On $X \times U_{rd}$ there is the following exact sequence of sheaves:

$$0 \to (\bigwedge^j \mathcal{O}^{\oplus q}) \otimes M_{r \times r} \xrightarrow{\exp} \mathrm{GL}(r, \mathcal{A}/(\mathrm{Nil})^{j+1}) \to \mathrm{GL}(r, \mathcal{A}/(\mathrm{Nil})^j) \to 0,$$

where $M_{r \times r}$ may be taken to be $\mathbb{C}^{r^2}$. Isomorphism classes of super vector bundles are given by $H^1(\mathrm{GL}(r, \mathcal{A}))$. The associated exact sequence of first cohomology for the above exact sequence of sheaves is

$$H^1((\bigwedge^j \mathcal{O}^{\oplus q})^{r^2} \otimes \mathcal{E}_{rd} \otimes \mathcal{E}^*_{rd}) \to H^1(\mathrm{GL}(r, \mathcal{A}/(\mathrm{Nil})^{j+1})) \to H^1(\mathrm{GL}(r, \mathcal{A})) .$$

The uniqueness of extension at each level of nilpotency is thus given by

$$H^1(\mathcal{O}^{\oplus \binom{q}{j} r^2} \otimes \mathcal{E}_{rd} \otimes \mathcal{E}^*_{rd}) .$$

But as stated before, $H^1(\mathcal{E}_{rd} \otimes \mathcal{E}^*_{rd}) = 0$ so that all the obstructions to uniqueness vanish.

3.2. DEFORMING SUBMANIFOLDS OF SUPERMANIFOLDS

In this subsection we show how a rigid classical submanifold $X^{r|0}$ of a supermanifold $Y^{n|m}$ may be deformed through a family of submanifolds each with the same normal bundle as $X^{r|0}$. This argument is a simplified version of LeBrun's work [14], which deforms a (not necessarily rigid) classical submanifold of a complex supermanifold. The more general work of deforming submanifolds $X^{r|p}$ in $Y^{n|m}$ has been done by Weintrob [21].

Let $X \subset Y$ be a compact complex submanifold of a complex manifold, and let $(Y, \mathcal{A})$ be a complex supermanifold. Let $\mathcal{I} \subset \mathcal{A}$ be the nilradical (i.e. the ideal of nilpotents) and let E be the bundle on Y defined implicitly by $\mathcal{O}(E^*) = \mathcal{I}/\mathcal{I}^2$. The normal bundle ν of $X \subset (Y, \mathcal{A})$ is by definition the graded bundle $\nu = \nu_0 \oplus \nu_1$, where $\nu_0 = (TY|_X)/TX$, and $\nu_1 = E|_X$.

Theorem 3.3 (LeBrun). *Suppose that*

$$H^1(X, \mathcal{O}(TX)) = H^1(X, \mathcal{O}(\nu)) = H^1(X, \mathcal{O}(\nu \otimes \nu^*)) = 0 .$$

Then there is a "complete, locally trivial, analytic family of submanifolds near X, biholomorphic to X and with normal bundle ν", whose tangent space at X is $H^0(X, \mathcal{O}(\nu))$. More precisely, there is a complex supermanifold $(W, \mathcal{B})$ of complex bidimension $(h^0(X, \mathcal{O}(\nu_0))|h^0(X, \mathcal{O}(\nu_1)))$, a submersive proper epimorphism

$$\pi : (S, \mathcal{C}) \to (W, \mathcal{B})$$

which is a fibering of complex supermanifolds, and a map of complex supermanifolds

$$\mu : (S, \mathcal{C}) \to (Y, \mathcal{A})$$

which is an embedding of $\pi^{-1}(t) \cong X$ into Y with normal bundle $\nu_t = \nu$ for all $t \in W$, such that $X = \mu(\pi^{-1}(x))$ for some $x \in W$ and such that the induced maps

$$T_x W \to H^0(X, \mathcal{O}(\nu_0)) , \qquad F_x \to H^0(X, \mathcal{O}(\nu_1))$$

are isomorphisms. Thus we assert the existence of ~~a~~ manifold $\overline{Z} = (Z, \mathcal{B})$ of bidimension $(h^0(TY/TX)|h^0(E))$, a supermanifold $\overline{F} = (F, \mathcal{C})$ of bidimension

$(h^0(TY/TX) + r|h^0(E))$, where $r = \dim X$, and a mapping diagram

$$\begin{tikzcd} & \overline{F} \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \overline{Y} & & \overline{Z} \end{tikzcd}$$

such that for some basepoint $z_0 \in Z$ one has $X = \beta_{rd}\alpha_{rd}^{-1}(z_0)$. $\overline{F}$ is a fibre bundle over $\overline{Z}$, with fibres X, such that the fibres embed into $\overline{Y}$ under β with normal bundle $\nu_t \cong \nu$ for all $t \in \overline{Z}$. Moreover, this family is universal in the sense that any diagram

$$\begin{tikzcd} & F_1 \arrow[dl] \arrow[dr] & \\ \overline{Y} & & \overline{Z}_1 \end{tikzcd}$$

is induced by a map $\overline{Z}_1 \to \overline{Z}$ in some neighborhood of the base point.

Proof. We begin by noticing that $H^1(X, TY/TX) = 0$ by hypothesis, so we may apply Kodaira's theorem [9]. This gives us a reduced family

$$\begin{tikzcd} & F \arrow[dl, "b"'] \arrow[dr, "a"] & \\ Y & & Z \end{tikzcd}.$$

Since $H^1(X, TX) = 0$ by hypothesis and the statement is local, let us assume $F = X \times Z$ where Z is a polydisk in $\mathbb{C}^{h^0(\nu_0)}$. Since $H^1(X, \nu \otimes \nu^*) = 0$, we have

$$H^1(X, \nu_0 \otimes \nu_0^*) = H^1(X, \nu_1 \otimes \nu_1^*) = 0 .$$

Thus, by lemma 3.1 of the previous section, $b^*E \cong \mathrm{pr}^* \nu_1$ and $b^*TY/TX \cong \mathrm{pr}^* \nu_0$ (i.e. the image of a fiber $X \times \{z\}$ in Y has normal bundle ν_0).

Let $\hat{E}^* \to Z$ be the vector bundle given by

$$\mathcal{O}(\hat{E}^*) = a_*^0(\mathcal{O}(b^*E)) \cong \mathcal{O}_Z^{\oplus h^0(\nu_1)}$$

by the Kunneth formula. Let $\mathcal{B} = \mathcal{O}(\bigwedge^\bullet \hat{E})$ and $\mathcal{C} = \mathcal{O}(\bigwedge^\bullet a^*\hat{E})$. The natural pull back map

$$a^{-1}\mathcal{O}(\bigwedge^\bullet \hat{E}) \to \mathcal{O}(a^* \bigwedge^\bullet \hat{E}) \cong \bigwedge^\bullet \mathcal{O}_F^{\oplus h^0(\nu_1)}$$

then defines a map $\alpha : \overline{F} \to \overline{Z}$, where $\overline{F} = (F, \mathcal{C})$ and $\overline{Z} = (Z, \mathcal{B})$. We now need to define a map $\beta : \overline{F} \to \overline{Y}$, i.e. a homomorphism $\beta^* : b^{-1}\mathcal{A} \to \mathcal{C}$. We build this in the following inductive way: let $\mathcal{N} \subset \mathcal{C}$ be the nilradical, and let $\mathcal{C}^{(m)} = \mathcal{C}/\mathcal{N}^{m+1}$. We then have the exact sequence of algebra homomorphisms:

$$0 \to \mathrm{Hom}_{\mathbb{C}}(b^{-1}\mathcal{A}, \bigwedge^m (\mathcal{N}/\mathcal{N}^2)) \to \mathrm{Hom}(b^{-1}\mathcal{A}, \mathcal{C}^{(m)})$$
$$\to \mathrm{Hom}(b^{-1}\mathcal{A}, \mathcal{C}^{(m-1)}) \to 0 .$$

But

$$\begin{aligned}\mathrm{Hom}_{\mathbb{C}}(b^{-1}\mathcal{A}, \textstyle\bigwedge^m (\mathcal{N}/\mathcal{N}^2)) &= \mathcal{O}(b^*E \otimes \textstyle\bigwedge^m a^*\hat{E}) \\ &= \mathcal{O}_F(b^*E) \otimes \textstyle\bigwedge^m \mathcal{O}_F^{\oplus h^0(\nu_1)}\end{aligned}$$

and

$$H^1(\mathcal{O}(b^*E \otimes \textstyle\bigwedge^m a^*\hat{E})) = H^1(\mathcal{O}_F(\mathrm{pr}^*\nu_1) \otimes \textstyle\bigwedge^m \mathcal{O}_F^{h^0(\nu_1)}) = 0$$

by the Kunneth formula and the assumption $H^1(\mathcal{O}(\nu_1)) = 0$. Hence every homomorphism extends. Finally, by lemma 3.2 of the previous section, $\beta^*(TY)/TX \cong \mathrm{pr}^*\nu$, so that we indeed do have a family of *normal* submanifolds.

Completeness of the family follows from exactly the same argument as given by Kodaira [9, pp. 158–160] building the map $\overline{Z}_1 \to \overline{Z}$ by higher and higher powers of the odd variables of $\overline{Z}_1$. Note that we need not be concerned about convergence since this is a power series in nilpotent variables which thus terminates. □

3.3. DEFORMING NORMAL QUADRICS

Now proceed in the opposite direction of the previous section, namely construct a superconformal manifold from its space of super light rays. We have the following:

Theorem 3.4. *If $\mathcal{N}^{5|2N}$ is a supermanifold with contact structure, then the space of "normal quadrics", that is, quadrics $Q_2 = \mathbb{P}_1 \times \mathbb{P}_1$, embedded with normal bundle*

$$\mathcal{O}(0,1) \otimes T^N \;\oplus\; \mathcal{O}(1,0) \otimes T^N \;\oplus\; T\mathbb{P}_3|_Q \otimes \mathcal{O}(-1,-1),$$

is a supermanifold $M^{4|4N}$ with superconformal structure. (Here, T^N denotes the N-dimensional trivial bundle.)

Proof. Let the contact structure of $\mathcal{N}$ be given by the line bundle valued one-form, θ. Let D be the kernel of θ. There is an exact sequence

$$0 \to D \to T\mathcal{N} \to L \to 0,$$

where L is the contact line bundle. L when restricted to a "normal quadric" is the $\mathcal{O}(1,1)$ line bundle.

The contact form is normal to each normal quadric since

$$j^*\theta \in H^0(Q, \Omega^1(L)) = H^0(Q, (\mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2)) \otimes \mathcal{O}(1,1)) = 0.$$

Thus $TQ \subset D|_Q$.

If we define $\mathcal{D} = D|_Q/TQ$ then we have the exact sequence

$$0 \to \mathcal{D} \to N \to L|_Q \to 0.$$

The exact sequence defining $\mathcal{D}$,

$$0 \to TQ \to D|_Q \to \mathcal{D} \to 0,$$

can be rewritten as

$$\begin{aligned} 0 &\to \mathcal{O}(2,0) \oplus \mathcal{O}(0,2) \to D|_Q \\ &\to \mathcal{O}(1,0) \otimes T \oplus \mathcal{O}(0,1) \otimes T^* \oplus \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \to 0. \end{aligned}$$

We can check that $H^1(Q, TQ \otimes \mathcal{D}^*) = 0$ and therefore this exact sequence splits: $D|_Q \cong TQ \oplus \mathcal{D}$.

Rewrite the first exact sequence, restricted to Q as

$$0 \to TQ \oplus \mathcal{D} \to T\mathcal{N}|_Q \to L|_Q \to 0$$

or

$$0 \to T_l Q \oplus T_r Q \oplus \eta_l \oplus \eta_r \oplus \nu_l \oplus \nu_r \to T\mathcal{N}|_Q \to L|_Q \to 0,$$

where $TQ_l = \mathcal{O}(2,0)$, $TQ_r = \mathcal{O}(0,2)$, $\eta_r = \mathcal{O}(1,-1)$, $\eta_l = \mathcal{O}(-1,1)$, $\nu_l = \mathcal{O}(1,0) \otimes T$, and $\nu_r = \mathcal{O}(0,1) \otimes T^*$. Consider

$$\Phi_{\mathcal{N}}|_Q : \bigwedge^2 (TQ \oplus \mathcal{D}) \to L|_Q,$$

where $\Phi_{\mathcal{N}} = [\ ,\ ]/D$ is the Frobenius form of $D \subset T\mathcal{N}$. Locally, $\Phi_{\mathcal{N}} = d\theta$ and is thus of full rank everywhere since $\theta \wedge (\theta)^{\wedge 2+N} \neq 0$ anywhere. We have

$$\begin{aligned} \Phi_{\mathcal{N}}|_Q &\in H^0(Q, (\Omega^2 Q \oplus \Omega^1 Q \otimes \mathcal{D}^* \oplus \textstyle\bigwedge^2 \mathcal{D}^*) \otimes L|_Q) \\ &= H^0(Q, \Omega^2 Q \otimes L|Q) \oplus H^0(Q, \Omega^1 Q \otimes \mathcal{D}^* \otimes L|_Q) \\ &\quad \oplus H^0(Q, \textstyle\bigwedge^2 \mathcal{D}^* \otimes L|_Q) \\ &= H^0(\mathcal{O}(-2,-2) \otimes \mathcal{O}(1,1)) \oplus H^0(\{(\mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2)) \\ &\quad \otimes (\mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O}(-1,0) \otimes T^* \oplus \mathcal{O}(0,-1) \otimes T) \otimes \mathcal{O}(1,1)\}) \\ &\quad \oplus H^0(\textstyle\bigwedge^2 \mathcal{D}^* \otimes L|_Q) \\ &= H^0(\mathcal{O}(-2,0) \otimes \mathcal{O}(1,-1) \otimes \mathcal{O}(1,1)) \\ &\quad \oplus H^0(\mathcal{O}(0,-2) \otimes \mathcal{O}(-1,1) \otimes \mathcal{O}(1,1)) \\ &\quad \oplus H^0(\mathcal{O}(1,-1) \otimes \mathcal{O}(-1,1) \otimes \mathcal{O}(1,1)) \\ &\quad \oplus H^0(\mathcal{O}(1,-1) \otimes \mathcal{O}(-1,0) \otimes T^* \otimes \mathcal{O}(1,1)) \\ &\quad \oplus H^0(\mathcal{O}(-1,1) \otimes \mathcal{O}(0,-1) \otimes T \otimes \mathcal{O}(1,1)) \\ &\quad \oplus H^0(\mathcal{O}(-1,0) \otimes T \otimes \mathcal{O}(0,-1) \otimes T^* \otimes \mathcal{O}(1,1)). \end{aligned}$$

Thus

$$\begin{aligned} \Phi_{\mathcal{N}}|_Q = {} & \Phi|_{T_l Q \otimes \eta_r} + \Phi|_{T_r Q \otimes \eta_l} + \Phi|_{\eta_l \otimes \eta_r} \\ & + \Phi|_{\eta_l \otimes \nu_r} + \Phi|_{\eta_r \otimes \nu_l} + \Phi|_{\nu_l \otimes \nu_r}. \end{aligned}$$

The first two terms are each nowhere zero, otherwise $\Phi_{\mathcal{N}}|_Q$ would not have full rank everywhere. The $\Phi|_{\nu_l \otimes \nu_r}$ must have full rank everywhere, otherwise we may take $\sigma \in \ker(\Phi|_{\nu_l \otimes \nu_r} : \nu_l \to \nu_r^* \otimes L|_Q)$, with $\sigma \neq 0$. Then $\sigma \sqcup \Phi|_{\eta_r \otimes \nu_l} = \sigma \sqcup \Phi|_{\mathcal{N}} \neq 0$ and $\sigma - (\sigma \sqcup \Phi|_{\eta_r \otimes \nu_l}) q_l$ is in the kernel of $\Phi_{\mathcal{N}}|_Q$. [Here, $q_l \in \Gamma(Q, TQ_l)$ is such that $\Phi_{\mathcal{N}}|_Q(q_l) = 1$ for some local trivialization of $\eta_r^* \otimes L|_Q$.] This contradicts $\Phi_{\mathcal{N}}$ having full rank everywhere.

Also note that

$$H^0(Q, \mathcal{O}(-1,0) \otimes T^* \otimes \mathcal{O}(-1,0) \otimes T^* \otimes \mathcal{O}(1,1)) = 0,$$
$$H^0(Q, \mathcal{O}(0,-1) \otimes T \otimes \mathcal{O}(0,-1) \otimes T \otimes \mathcal{O}(1,1)) = 0.$$

Hence $\Phi|_{\Lambda^2 \nu_l} = \Phi|_{\Lambda^2 \nu_r} = 0$.

Now consider the long exact sequence

$$\begin{aligned} 0 &\to H^0(Q, \mathcal{D}) \to H^0(Q, N) \to H^0(Q, L|_Q) \\ &\to H^1(Q, \mathcal{D}) \to H^1(Q, N) \to H^1(Q, L|_Q) \to \cdots . \end{aligned}$$

Since

$$H^1(Q, L|_Q) = H^1(Q, \mathcal{O}(1,1)) = 0$$

and

$$\begin{aligned} H^1(Q, \mathcal{D}) &= H^1(Q, \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1) \oplus \mathcal{O}(1,0) \otimes T \oplus \mathcal{O}(0,1) \otimes T^*) \\ &= 0, \end{aligned}$$

we can conclude that $H^1(Q, N) = 0$.

We leave it to the reader to show that $H^1(Q, N \otimes N^*) = 0$. We also note here that

$$H^1(Q, TQ) = H^1(Q, \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)) = 0.$$

By the deformation theory examined in the first subsection, the space of normal quadrics is then a supermanifold, M, with $TM_Q \cong H^0(Q, N)$, and $\dim(TM) = \dim(H^0(Q, N)) = 4|4N$. We also have the total space of this family of quadrics, $F^{6|4N}$ and the diagram:

$$\begin{tikzcd} & F \arrow[dl, "\rho"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & M \end{tikzcd}$$

where the dimensions of the fibres of ρ and π are respectively $1|2N$ and 2. The fibres are also transverse to each other. F is then a $\mathbb{P}_1 \times \mathbb{P}_1$ fibration over M.

Now $F \overset{\text{graph}(\rho,\pi)}{\hookrightarrow} \mathcal{N} \times M$. We thus have the exact sequence

$$0 \to TF \to \rho^* T\mathcal{N} \oplus \pi^* TM \to N_F \to 0.$$

Let $TQ \equiv TF/M \equiv \ker(\pi_* : TF \to \pi^* TM)$. We then have, since the fibres of ρ and π are transverse to each other, that $TQ \subset \rho^* T\mathcal{N}$ and hence

$$0 \to TQ \xrightarrow{\rho_*} \rho^* T\mathcal{N} \to N \to 0,$$

where $N \equiv \rho^* T\mathcal{N}/TQ$.

We also have

$$0 \to \rho^* D \to \rho^* T\mathcal{N} \to \rho^* L \to 0.$$

Let U be a small enough polydisk in M so that we have

$$\pi^{-1}(U) \cong (\mathbb{P}_1 \times \mathbb{P}_1) \times U.$$

With such an identification we have the projection

$$\mathbf{pr} : \mathbb{P}_1 \times \mathbb{P}_1 \times U \to \mathbb{P}_1 \times \mathbb{P}_1 .$$

Using lemma 3.2, we can then write

$$\begin{aligned} N &\cong \mathbf{pr}^* \mathcal{O}(1,0) \otimes T \oplus \mathbf{pr}^* \mathcal{O}(0,1) \otimes T^* \oplus \mathbf{pr}^* T\mathbb{P}_3|_{Q_2 \cong \mathbb{P}_1 \times \mathbb{P}_1}, \\ TQ &\cong \mathbf{pr}^* \mathcal{O}(2,0) \oplus \mathbf{pr}^* \mathcal{O}(0,2), \end{aligned}$$

and $\rho^* L \cong \mathbf{pr}^* \mathcal{O}(1,1)$.

As before, we have $\rho^* \theta|_{TQ} = 0$ and thus $TQ \hookrightarrow \rho^* D$. This gives

$$0 \to \mathcal{D} \to N \to \rho^* L \to 0,$$

where $\mathcal{D} \equiv \rho^* D/TQ$. Using lemma 3.2, we will also have

$$\mathcal{D} \cong \mathbf{pr}^* (\mathcal{O}(1,0) \otimes T \oplus \mathcal{O}(0,1) \otimes T^* \oplus \mathcal{O}(1,-1) \oplus \mathcal{O}(-1,1)) \ .$$

Note that we also have that the exact sequence defining $\mathcal{D}$ splits so that $D \cong TQ \oplus \mathcal{D}$. [From the above one may define $\eta_l = \mathbf{pr}^* \mathcal{O}(1,-1)$, $\eta_r = \mathbf{pr}^* \mathcal{O}(-1,1)$, $\nu_l = \mathbf{pr}^* \mathcal{O}(1,0) \otimes T$, and $\nu_r = \mathbf{pr}^* \mathcal{O}(0,1) \otimes T^*$ with of course $\mathcal{D} = \eta_l \oplus \eta_r \oplus \nu_l \oplus \nu_r$.]

We have from the exact sequence

$$0 \to \mathcal{D} \to N \to \rho^* L \to 0$$

and writing $Q = \mathbb{P}_1 \times \mathbb{P}_1$, the long exact sequence

$$\begin{aligned} 0 &\to H^0(Q \times U, \mathbf{pr}^* (\mathcal{O}(1,0) \otimes T)) \oplus H^0(Q \times U, \mathbf{pr}^* (\mathcal{O}(0,1) \otimes T^*)) \\ &\to H^0(\pi^{-1}(U), N) \to H^0(Q \times U, \mathbf{pr}^* (\mathcal{O}(1,1))) \\ &\to H^1(Q \times U, \mathbf{pr}^* (\mathcal{O}(1,0) \otimes T)) \oplus H^1(Q \times U, \mathbf{pr}^* (\mathcal{O}(0,1) \otimes T^*)) \to \cdots . \end{aligned}$$

Applying the Kunneth formula and the fact that U is a polydisk, the last two terms written are zero. Hence there is the exact sequence of sheaves over M,

$$0 \to S_+ \otimes E \oplus S_- \otimes E^* \to TM \to S_+ \otimes S_- \to 0,$$

where

$$S_+(U) = H^0(Q \times U, \mathbf{pr}^* (\mathcal{O}(1,0))), \qquad S_-(U) = H^0(Q \times U, \mathbf{pr}^* (\mathcal{O})),$$

and $E(U) = H^0(Q \times U, \mathbf{pr}^* (T))$. Writing $T_l M \equiv S_+ \otimes E$, $T_r M \equiv S_- \otimes E^*$, and $T_0 M \equiv S_+ \otimes S_-$, this exact sequence is

$$0 \to T_l M \oplus T_r M \to TM \to T_0 M \to 0.$$

The Frobenius form $\Phi_M : \bigwedge^2 (T_l M \oplus T_r M) \to T_0 M$ is defined by

$$\Phi_M(X,Y) = [X,Y]/T_l M \oplus T_r M$$

for $X \in \Gamma(T_l M)$ and $Y \in \Gamma(T_r M)$. We wish to show

$$\Phi_M|_{\bigwedge^2 T_l M} = \Phi_M|_{\bigwedge^2 T_r M} = 0$$

and that $\Phi_M|_{\bigwedge^2 T_l M \otimes T_r M}$ corresponds to the convolution

$$S_+ \otimes E \otimes E^* \otimes S_- \to S_+ \otimes S_-,$$

in order to show that M has a superconformal structure induced from $\mathcal{N}$.

We have

$$\begin{array}{ccccccccc} 0 & \to & TQ & \to & TF & \xrightarrow{\pi_*} & \pi^* TM & \to & 0 \\ & & & & \downarrow \rho_* & & \downarrow \rho_* & & \\ 0 & \to & TQ & \xrightarrow{\rho_*} & \rho^* T\mathcal{N} & \xrightarrow{q_*} & N & \to & 0 \end{array}$$

The map ρ_* actually provides an isomorphism between $\rho^{-1} TM(U \times Q)$ and $H^0(U \times Q, N)$ for U a small enough open set in M. We may also assume that $F = U \times Q$ and thus that $TF = TQ \oplus TM$. We have $\rho_*[X,Y] = [\rho_* X, \rho_* Y]$ for $X, Y \in \Gamma(\rho^{-1} TM)$. Thus

$$[T_l M, T_l M] \bmod T_l M \oplus T_r M$$

corresponds to

$$[\nu_l + TQ, \nu_l + TQ] \bmod D.$$

This is just $\rho^* \Phi_{\mathcal{N}}|_{(\nu_l + TQ) \otimes (\nu_l + TQ)}$, which we have already calculated to be zero. We can conclude that

$$[T_l M, T_l M] \subset T_l M \oplus T_r M.$$

Now consider $[T_l M, T_l M] \bmod T_l M$. Under ρ_* this corresponds to

$$[\nu_l + TQ, \nu_l + TQ] \bmod \nu_l \oplus TQ.$$

This represents a section of

$$H^0(U \times Q, (\bigwedge^2 (\nu_l^* \oplus T^* Q)) \otimes (\eta_l \oplus \eta_r \oplus \nu_r)).$$

Since

$$\nu_l \cong \mathrm{pr}^* \mathcal{O}(1,0) \otimes T, TQ \cong \mathrm{pr}^* \mathcal{O}(2,0) \oplus \mathrm{pr}^* \mathcal{O}(0,2),$$
$$D/(\nu_l \oplus TQ) \cong \mathrm{pr}^* \mathcal{O}(1,-1) \oplus \mathrm{pr}^* \mathcal{O}(-1,1) \oplus \mathrm{pr}^* (0,1) \otimes T^*,$$

we have that this cohomology group is zero. $T_l M$ is thus an integrable distribution. Similarly, $T_r M$ is integrable.

Also note that if $X \in \Gamma(T_l M), X \neq 0$ then under the correspondence given by ρ_*,

$$\Phi_M|_{T_l M \otimes T_r M}(X, \bullet) = \rho^* \Phi_{\mathcal{N}}|_{\nu_l \otimes \nu_r}(X, \bullet) \neq 0$$

and similarly for $Y \in \Gamma(T_r M)$.

Thus

$$\begin{aligned}\Phi_M|_{T_l M\otimes T_r M} &\in H^0(Q\times U, \mathcal{O}(-1,0)\otimes T\otimes\mathcal{O}(0,-1)\otimes T^*\otimes\mathcal{O}(1,1))\\ &= H^0(Q\times U, T\otimes T^*)\end{aligned}$$

with full rank and by the definitions of S_+, S_- and E, we see that Φ_M acts via the contraction map

$$S_+\otimes E\otimes E^*\otimes S_- \to S_+\otimes S_-. \qquad \square$$

4. Extending conformal structures

4.1. THICKENINGS AND POISSON STRUCTURES

We present in this subsection the definition of *thickenings* of complex manifolds given in Eastwood and LeBrun [6]. We will also present the definition of a *Poisson thickening* given in LeBrun [13].

Let X be a complex manifold. A *thickening* of order m, $X_{(m)}$, of X is a ringed space, $(X, \mathcal{O}_{(m)})$, where $\mathcal{O}_{(m)}$ is a sheaf of $\mathbb{C}$-algebras, locally isomorphic to $\mathcal{O}(t)/t^{m+1}$, and which satisfies $\mathcal{O}_{(m)}/\,\mathrm{Nil}\cong\mathcal{O}$, where Nil denotes the subsheaf of nilpotents in $\mathcal{O}_{(m)}$. The tangent bundle of $X_{(m)}$ may be defined as the sheaf

$$TX_{(m)} = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{(m)}, \mathcal{O}_{(m)})$$

and the cotangent bundle may be defined as the sheaf

$$\Omega^1 X_{(m)} = \mathrm{Hom}(TX_{(m)}, \mathcal{O}_{(m)}).$$

Now let X be a complex contact manifold. Let L be its contact line bundle. The total space of $L - 0_L$ has the structure of a *Poisson manifold*, i.e., it is equipped with a global bivector field τ given locally by

$$\tau = t\left[\left(t\frac{\partial}{\partial t} + \Sigma p_j\frac{\partial}{\partial p_j}\right)\wedge\frac{\partial}{\partial q^0} + \Sigma\frac{\partial}{\partial q^j}\wedge\frac{\partial}{\partial p_j}\right],$$

where t is the fiber coordinate on L and the other coordinates are contact coordinates lifted from X. τ defines a Poisson bracket on L,

$$\{\ ,\ \}:\mathcal{O}\to\mathcal{O}$$

given by $\{f,g\} = \tau(df,dg)$.

Let $\mathcal{T}\subset\mathcal{O}$ denote the ideal of functions vanishing on $X = 0_L\subset L$. We have

$$\{\mathcal{T}^k, \mathcal{T}^l\}\subset\mathcal{T}^{k+l}.$$

If we define $\mathcal{O}_m = \mathcal{O}/\mathcal{T}^{m+1}$, then $\{\ ,\ \}$ gives $\mathcal{O}_m$ the structure of a sheaf of nilpotent Lie algebras. Moreover, since $\mathbb{C}$ is contained in the center (with respect to $\{\ ,\ \}$) of $\mathcal{O}_m$, $\mathcal{O}_m/\mathbb{C}$ becomes a sheaf $\mathcal{A}_{m+1}$ of nilpotent Lie algebras. We define

$$\mathcal{G}_m := \exp\mathcal{A}_m,$$

thereby obtaining a sheaf of nilpotent Lie groups. Now there is a natural injective map

$$\mathcal{O}_{m-1}/\mathbb{C} \to \mathrm{Der}(\mathcal{O}_m)$$

given by $f \mapsto \{f, \cdot\}$ and this realizes $\mathcal{A}_m$ as a nilpotent subalgebra of $\mathrm{Der}(\mathcal{O}_m)$. Therefore $\mathcal{G}_m$ is a nilpotent subgroup of $\mathrm{Aut}(\mathcal{O}_m)$.

Isomorphism classes of thickenings of X are precisely given by elements of $H^1(\mathrm{Aut}(\mathcal{O}_m))$. We have therefore the following definition of a *Poisson thickening*: A thickening of X of order m is said to be a Poisson thickening if its isomorphism class is in the image of

$$H^1(X, \mathcal{G}_m) \to H^1(X, \mathrm{Aut}(\mathcal{O}_m)).$$

4.2. "SUPERFYING" AMBITWISTORS

We now show that every space of null geodesics can be imbedded in a supermanifold of dimension $5|2m$, for $m \leq 4$. Let $\mathcal{N}^5$ be a space of null geodesics for some spacetime M^4. LeBrun [13] has shown that $\mathcal{N}$ has an extension to a Poisson thickening, $\mathcal{N}^{(m)}$, of order m for $m \leq 4$. If the Bach tensor of M^4 vanishes, then $\mathcal{N}$ has an extension to a Poisson thickening of order $m = 5$. If the Eastwood–Dighton tensor of M^4 vanishes then $\mathcal{N}$ has an extension to a Poisson thickening of order $m = 6$.

LeBrun also constructs a supermanifold $\mathcal{N}^{5|2m}$ from $\mathcal{N}^{(m)}$. Let us recall this construction. It is (p. 66 of LeBrun [13]):

Let $\mathcal{O}_{(m)}(1,1)$ be the "divisor line bundle" of $\mathcal{N} \subset \mathcal{N}^{(m)}$. The line bundle $\mathcal{O}_{(m)}(1,1)$ has a canonical section σ vanishing along $\mathcal{N}$. Let $\mathcal{O}_{(m)}(0,1)$ and $\mathcal{O}_{(m)}(1,0)$ be extensions of $L_\pm$ to $\mathcal{N}^{(m)}$, and let $\mathcal{T}$ be a complex vector space of dimension m. Then

$$\mathcal{O}_{(m)}(1,1) \oplus \bigwedge^2 [\mathcal{T} \otimes \mathcal{O}_{(m)}(-1,0) \oplus \mathcal{T}^* \otimes \mathcal{O}_{(m)}(0,-1)] \otimes \mathcal{O}_{(m)}(1,1)$$

has a canonical section $\hat{\sigma} = \sigma + \mathrm{id}$ where

$$\mathrm{id} \in \mathcal{T} \otimes \mathcal{T}^* \subset \bigwedge^2 (\mathcal{T} \oplus \mathcal{T}^*).$$

$\hat{\sigma}$ generates an even ideal $\mathcal{J}$ in

$$\bigwedge^\bullet [\mathcal{T} \otimes \mathcal{O}_{(m)}(-1,0) \otimes \mathcal{T}^* \otimes \mathcal{O}_{(m)}(0,-1)],$$

i.e., for every local trivialization of $\mathcal{O}_{(m)}(1,1)$ $\hat{\sigma}$ gives a section of this bundle and changing trivialization just multiplies this section by an element of $\mathcal{O}_{(m)}$. Thus

$$\mathcal{N}^{[m]} = (\mathcal{N}, \bigwedge^\bullet [\mathcal{T} \otimes \mathcal{O}_{(m)}(-1,0) \oplus \mathcal{T}^* \otimes \mathcal{O}_{(m)}(0,-1)]/\mathcal{J})$$

is a well defined $\mathbb{Z}_2$-graded complex ringed space. Moreover $\mathcal{N}^{[m]}$ is a complex supermanifold, i.e., it is locally isomorphic to $\mathcal{O}(\bigwedge^\bullet \mathbb{C}^{2m})$. The nilpotents of $\mathcal{O}_{(m)}$ have become the nilpotents of $\bigwedge^\bullet (\mathcal{T} \otimes \mathcal{T}^*)$!

4.3. THE CONTACT STRUCTURE OF $L^*_{+(m)}$

We shall first show that a contact structure exists on the total space of the line bundle $L^*_{+(m)}$, and then we will be able to show in the next section how this "descends" to our supermanifold $\mathcal{N}^{5|2m}$.

We may locally lift a set of Darboux coordinates, q^j, p_j, on $\mathcal{N}$ to a set of coordinates q^j, p_j, t on $\mathcal{N}_{(m)}$. Let $f_{\alpha\beta}$ be such that $\exp(\tau \sqcup df_{\alpha\beta})$ is the change of coordinates on $\mathcal{N}_{(m)}$ between two open sets U_α and U_β. Here τ is the *exelissic form* given by

$$\tau = t\left[\left(t\frac{\partial}{\partial t} + \Sigma p_j \frac{\partial}{\partial p_j}\right) \wedge \frac{\partial}{\partial q^0} + \Sigma \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}\right].$$

We have on a coordinate neighborhood, U_β, the one-form

$$\theta_\beta = dq^0_\beta + p_{\beta j} dq^j_\beta .$$

Consider how this changes under a coordinate transformation, i.e.,

$$\exp(\tau \sqcup df_{\alpha\beta})^* \theta_\beta .$$

If we write $X_{\alpha\beta} \cong \tau \sqcup df_{\alpha\beta}$ then

$$\exp(\tau \sqcup df_{\alpha\beta})^* \theta_\beta = \exp(\mathcal{L}_{X_{\alpha\beta}}) \theta_\beta .$$

We have (dropping the use of the subscripts α and β) that

$$\mathcal{L}_X \theta = -t \frac{\partial f}{\partial q^0} \theta + td\left(f + t\frac{\partial f}{\partial t}\right).$$

We have then:

Claim 4.1.

$$\exp(\mathcal{L}_X)\theta = \frac{1}{t}\left(\sum_{k=0}^{N} X^k(t)\right)\theta + \left(\sum_{k=0}^{N-1} X^k(t)\right) d\left(\sum_{k=1}^{N-1} \frac{X^{k-1}}{k!}(f + t\frac{\partial f}{\partial t})\right) \bmod t^{N+1},$$

i.e.,

$$\exp(\tau \sqcup df)^*\theta = \left(\frac{1}{t}\exp(X)(t)\right)\theta + \exp(X)(t) d(\mathcal{F}_+) \bmod t^{N+1},$$

where

$$\mathcal{F}_+ = \sum_{k=1}^{N-1} \frac{X^{k-1}}{k!}\left(f + t\frac{\partial f}{\partial t}\right).$$

Note that if $f^{(0)}$ has homogeneity zero in t then

$$\frac{f^{(0)}}{t} + t\frac{\partial}{\partial t}\left(\frac{f^{(0)}}{t}\right) = 0.$$

To prove the claim, one may first prove by induction

$$\mathcal{L}_X^N \theta = \frac{1}{t} X^N(t)\theta + \sum_{k+j=N-1} \frac{X^k(t)}{k!}\frac{dX^j g}{(j+1)!} N!,$$

where $g = f + t\partial f/\partial t$. This proof is left to the reader.

We thus see that

$$\sum_{k=0}^{N} \frac{1}{k!}\mathcal{L}_X^k \theta = \frac{\exp X(t)}{t}\theta + \exp X(t) d\mathcal{F}_+ \bmod t^{N+1}.$$

Hence on an overlap of two open sets $U_\alpha \cap U_\beta$, we have

$$(\delta\theta)_{\alpha\beta} = \theta_\alpha - \frac{t}{\exp X_{\alpha\beta}(t)}(\exp X_{\alpha\beta})^* \theta_\beta = t d\mathcal{F}_{+\alpha\beta}.$$

Thus $\delta(td\mathcal{F}_+) = td(\delta\mathcal{F}_+) = 0$. We see that

$$(\delta\mathcal{F}_+)_{\alpha\beta\gamma} \bmod t^m = c,$$

a constant on triple overlaps. We consider the part of this equation with zero homogeneity in t. This constant must be cohomologous to an integer, since the left hand side is now $(\delta f_+)_{\alpha\beta\gamma}$ where $\exp(f_{+\alpha\beta})$ are the transition functions of the line bundle L_+. We conclude that $\exp(\mathcal{F}_{+\alpha\beta})$ form transition functions for a line bundle over $\mathcal{N}_{(m-1)}$ which is an extention of L_+. There is already a unique extension, $L_{(m)+}$ of L_+ over $\mathcal{N}_{(m)}$, which gives a unique extension over $\mathcal{N}_{(m-1)}$. [Recall that L_+ is just notation for $\mathcal{O}(0,1)$.] Thus we may extend $\exp(\mathcal{F}_{+\alpha\beta})$ to be transition functions for $L_{(m)+}$.

Let $\{\mathbf{0}_{L^*_{(m)+}}\}$ denote the zero section of $L^*_{(m)+}$. One may now check that the twisted one-form on $L^*_{(m)+} - \{\mathbf{0}_{L^*_{(m)+}}\}$, given locally by $\theta - ts_+^{-1}ds_+$, where s_+ is the "coordinate along the fiber", gives a contact structure on $L^*_{(m)+} - \{\mathbf{0}_{L^*_{(m)+}}\}$ with the contact line bundle being the pull back of $\mathcal{O}_{(m)}(1,1)$ from $\mathcal{N}_{(m)}$. Henceforth we write $\mathcal{L}^*_{+(m)}$ for $L^*_{(m)+} - \{\mathbf{0}_{L^*_{(m)+}}\}$.

4.4. THE SUPERCONTACT STRUCTURE

We now show how the contact structure constructed in the previous subsection will "descend" to our supermanifold $\mathcal{N}^{5|2m}$. Consider the *superthickening*

$$\mathcal{L}^*_{+(m)[m]} = (\mathcal{L}^*_{+(m)}, \bigwedge^{\bullet}(\mathcal{O}_{(m)}(-1,0)\otimes\mathcal{T} \oplus \mathcal{T}^*\otimes\mathcal{O}_{(m)}(0,-1))).$$

Recall that $\mathcal{O}_{(m)}(0,-1) \cong \mathcal{O}_{(m)}$ on $\mathcal{L}^*_{+(m)}$. Choose m linearly independent sections

$$e^i \in \Gamma(\mathcal{O}_{(m)}(1,0)\otimes\mathcal{O}_{(m)}(-1,0)\otimes\mathcal{T}) \subset \Gamma(\mathcal{O}_{(m)[m]}(1,0))$$

and m linearly independent sections dual to the above,

$$e_i \in \Gamma(\mathcal{O}_{(m)}(0,1) \otimes \mathcal{O}_{(m)}(0,-1) \otimes \mathcal{T}^*) \subset \Gamma(\mathcal{O}_{(m)[m]}(0,1)).$$

Note that

$$\mathcal{O}_{(m)[m]}(0,1) \cong \mathcal{O}_{(m)[m]}$$

on $\mathcal{L}^*_{+(m)[m]}$ so that $e^i de_i$ makes sense as a global section of

$$\mathcal{O}_{(m)[m]}(1,1) \otimes \Omega^1 \mathcal{L}^*_{+(m)[m]}.$$

(Note that $\mathcal{O}_{(m)[m]}(1,1) \cong \mathcal{O}_{(m)[m]}(1,0))$ on $\mathcal{L}^*_{+(m)[m]}$.)

Let s_+ be the coordinate along the fiber of $\mathcal{L}^*_{+(m)}$ and s_- a local section of $\mathcal{O}_{(m)}(-1,0)$. Also let $\phi^i \equiv s_- e^i$ and $\psi_i \equiv s_+ e_i$ be the odd coordinates on $\mathcal{L}^*_{+(m)[m]}$. Since $\mathcal{L}^*_{+(m)[m]}$ is split, $\theta - ts_+^{-1} ds_+$ is a well defined twisted one-form on it. We have then

$$\begin{aligned}
&\theta - ts_+^{-1} ds_+ + s_-^{-1}(s_- e^i) d(s_+^{-1} s_+ e_i) \\
&= \theta - tds_+^{-1} ds_+ + s_-^{-1}\phi^i d(s_+^{-1}\psi_i) \\
&= \theta - ts_+^{-1} ds_+ - s_-^{-1} s_+^{-2}\phi^i \psi_i ds_+ + s_-^{-1}\phi^i s_+^{-1} d\psi_i \\
&= \theta - ts_+^{-1} ds_+ - s_-^{-1} s_+^{-1}\phi^i \psi_i s_+^{-1} ds_+ + s_-^{-1} s_+^{-1}\phi^i d\psi_i .
\end{aligned}$$

When pulled back to $\mathcal{L}^*_{+[m]} = \{t + s_-^{-1} s_+^{-1}\phi^i\psi_i = 0\}$ this is $\theta + s_-^{-1} s_+^{-1}\phi^{-1} d\psi_i$ and thus descends to an $\mathcal{O}_{[m]}(1,1)$-valued contact one-form on $\mathcal{N}_{[m]}$.

4.5. EXTENDING CONFORMAL STRUCTURES

A complex conformal spacetime is said to be *civilized* if its space of null geodesics forms a complex manifold. It is said to be *reflexive* if it is the space of normal quadrics for its space of null geodesics.

Corollary 4.2. *Let M^4 be a complex conformal manifold. Assume M is civilized and reflexive. M then has an extension to a complex superconformal manifold $M^{4|4m}$, $m \leq 4$; if the Bach tensor vanishes M has an extension to a superconformal manifold, $M^{4|20}$; if the Eastwood–Dighton tensor vanishes, M has an extension to a superconformal manifold, $M^{4|24}$.*

In general, if the ambitwistor space $\mathcal{N}^5$ has a Poisson thickening of order m, then M^4 may extended to a superconformal manifold $M^{4|4m}$.

Proof. Let $\mathcal{N}$ be the ambitwistor space of M. By our assumptions for each m and our previous results, $\mathcal{N}^5$ has an extension to a supercontact manifold $\mathcal{N}^{5|2m}$. Since M is reflexive, it is the reduced space of the space $M^{4|4m}$ of normal quadrics in $\mathcal{N}^{5|2m}$. $M^{4|4m}$ by its construction is a superconformal manifold. □

(Note: For $m \leq 4$, there were no special assumptions, beyond civility and reflexivity, on our spacetime M^4.)

This result is now a fully curved version of the linearized result of Chau and Lim [5].

5. $N = 3$ SSYM equations and integrability

5.1. INTEGRABILITY ALONG SUPER LIGHT RAYS

Recall that a superconformal structure is partly given by the exact sequence:

$$0 \to S_+ \otimes E \oplus S_- \otimes E^* \to TM \to S_+ \otimes S_- \to 0 .$$

Choose a local splitting of this exact sequence so that

$$TM \cong S_+ \otimes S_- \oplus S_+ \otimes E \oplus S_- \otimes E^* .$$

We assume that our connection is given (locally) by

$$d + \mathcal{A} = d + (A_{\alpha\dot{\alpha}}, \omega_{\alpha i}, \omega^{j}_{\dot{\alpha}}) .$$

Integrability of this connection along super light rays is by definition the vanishing of the curvature of this connection when it is restricted to a super light ray. This implies that the curvature has a special form. Consider the (local) decomposition of $\Omega^2 M$ as

$$\begin{aligned}\Omega^2 M \cong\ & \bigwedge^2 (S_+^* \otimes E^*) \oplus \bigwedge^2 (S_-^* \otimes E) \oplus \bigwedge^2 (S_+^* \otimes S_-^*) \\ & \oplus S_+^* \otimes S_-^* \otimes S_+^* \otimes E^* \oplus S_+^* \otimes S_-^* \otimes S_-^* \otimes E \oplus S_+^* \otimes E^* \otimes S_-^* \otimes E \\ =\ & \bigwedge^2 S_+^* \otimes \bigwedge^2 \pi E^* \oplus \odot^2 S_+^* \otimes \odot^2 \pi E^* \oplus \bigwedge^2 S_-^* \otimes \bigwedge^2 \pi E \\ & \oplus \odot^2 S_-^* \otimes \odot^2 \pi E \oplus \bigwedge^2 S_-^* \otimes \odot^2 S_+^* \oplus \bigwedge^2 S_+^* \otimes \odot^2 S_-^* \\ & \oplus \bigwedge^2 S_+^* \otimes S_-^* \otimes E^* \oplus \odot^2 S_2^* \otimes S_-^* \otimes E^* \\ & \oplus \bigwedge^2 S_-^* \otimes S_+^* \otimes E \oplus \odot^2 S_-^* \otimes S_+^* \otimes E \oplus S_+^* \otimes E^* \otimes S_-^* \otimes E .\end{aligned}$$

The tangent space of a super light ray is generated by super light vectors which are of the form

$$\eta^\alpha \otimes \nu^{\dot{\beta}} + \eta^\alpha \otimes e^i + \nu^{\dot{\beta}} \otimes e_j ,$$

where the η^α and $\nu^{\dot{\beta}}$ are fixed sections of S_+ and S_- (except for scaling), and the e^i and e_j are sections of E and E^* that are allowed to vary freely. The vanishing of the curvature F_{AB} on the super light ray implies for example that $F_{AB}(\eta^\alpha \otimes e^i, \eta^\beta \otimes e^j) = 0$, i.e., F_{AB} has no component in $\odot^2 S_+^* \otimes \odot^2 E^*$, and similarly for other components.

We thus obtain

$$F_{AB} \in \bigwedge^2 S_+^* \otimes \bigwedge^2 \pi E^* \oplus \bigwedge^2 S_-^* \otimes \bigwedge^2 \pi E \oplus \bigwedge^2 S_+^* \otimes \odot^2 S_-^*$$
$$\oplus \bigwedge^2 S_-^* \otimes \odot^2 S_+^* \oplus \bigwedge^2 S_+^* \otimes S_-^* \otimes E^* \oplus \bigwedge^2 S_-^* \otimes S_+^* \otimes E .$$

Thus

$$F_{AB} = W_{ij}\epsilon_{\alpha\beta} + W^{ij}\epsilon_{\dot\alpha\dot\beta} + f_{\dot\alpha\dot\beta}\epsilon_{\alpha\beta} + f_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} + \chi_{\dot\alpha i}\epsilon_{\alpha\beta} + \chi_{\alpha}^{j}\epsilon_{\dot\alpha\dot\beta} .$$

(This notation now coincides with Harnad et al. [7].)

5.2. EXTERIOR DERIVATIVES AND CONNECTIONS

We shall now define (at least locally) a certain operator on $\Omega^p M$; it is an "exterior derivative", Δ, that is similiar to the regular exterior derivative, d, but such that $\Delta^2 \neq 0$ in general. The $N = 3$ SSYM equations will be written in terms of components of Δ. Δ actually comes from the nonintegrability of $T_l M \oplus T_r M$.

Once again, consider a local splitting of the exact sequence

$$0 \to \Omega_0^1 M \to \Omega^1 M \to \Omega_l^1 M \oplus \Omega_r^1 M \to 0$$

so that

$$\Omega^1 M \cong \Omega_0^1 M \oplus \Omega_l^1 M \oplus \Omega_r^1 M .$$

The Frobenius form $\Phi : \Omega_0^1 M \to \Omega_l^1 M \otimes \Omega_r^1 M$ is then well defined (locally) as a map from $\Omega_0^1 M$ to $\Omega^2 M$. Define $\Delta : \Omega_0^1 M \to \Omega^2 M$ by

$$\Delta = d - \Phi .$$

On $\Omega_l^1 M \oplus \Omega_r^1 M$ define $\Delta : \Omega_l^1 M \oplus \Omega_r^1 M \to \Omega^2$ to be $\Delta \equiv d$. Also define $\Delta f \equiv df$ for superfunctions f.

Now extend Δ to all of $\Omega^\bullet M$ by the Leibnitz rule:

$$\Delta(\omega_1 \wedge \omega_2) \equiv (\Delta\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge \Delta\omega_2 .$$

We may consider connections on vector bundles coming from this "exterior differentiation", $D : \Gamma(E \otimes \Omega^p M) \to \Gamma(E \otimes \Omega^{p+1} M)$, where

$$D(\sigma \otimes \omega) = D(\sigma) \wedge \omega + \sigma \otimes \Delta(\omega)$$

for $\sigma \in \Gamma(E)$ and $\omega \in \Gamma(\Omega^\bullet M)$.

Let $\Delta_{\alpha i} = \pi_l \circ \Delta$, $\Delta_{\dot\alpha}^{j} = \pi_r \circ \Delta$ and $\Delta_{\alpha\dot\alpha} = \pi_0 \circ \Delta$.

Proposition 5.1. $[\Delta_{\alpha i}, \Delta_{\dot\alpha}^{j}] = -\Phi_{\alpha\dot\alpha i}^{\beta\dot\beta j}\Delta_{\beta\dot\beta}$.

Consider Φ as an operator on $\Omega^\bullet M$ by $\Phi \equiv 0$ on $\Omega_l^1 M$ and $\Omega_r^1 M$, $\Phi f \equiv 0$ for $f \in \Gamma(A)$, a superfunction, and extend to all of $\Omega^\bullet M$ by the Leibnitz rule.

From this, it is clear that $d = \Delta + \Phi$ on all of $\Omega^\bullet M$ and that $\Phi^2 = 0$. Since $d^2 = 0$ we have $(\Delta + \Phi)^2 = 0$ and thus

$$\Delta^2 = -\Phi\Delta - \Delta\Phi$$

or

$$(\Delta_{\alpha\dot\alpha} + \Delta_{\alpha i} + \Delta^{j}_{\dot\alpha})^2 = -\Phi\Delta - \Delta\Phi\,.$$

Let $\kappa_A \in \Gamma(\Omega^p M)$, $|A| = p$, where A is a multi-index, and elements of A are indices of the form $(\alpha\dot\alpha)$, (αi), and $(^{j}_{\dot\alpha})$. Consider both sides of

$$(\Delta_{\alpha\dot\alpha} + \Delta_{\alpha i} + \Delta^{j}_{\dot\alpha})^2\kappa_A = -(\Phi\Delta + \Delta\Phi)\kappa_A$$

and the terms in each which have values in $\Omega^p_A M \cdot \Omega^1_l M \cdot \Omega^1_r M$. We also assume that Φ corresponds to convolution so that $\Phi^{\gamma\dot\gamma j}_{\alpha\dot\alpha i} = \delta^\gamma_\alpha \delta^{\dot\gamma}_{\dot\alpha} \delta^j_i$ and thus $\Delta_{\alpha\dot\alpha}(\Phi^{\gamma\dot\gamma j}_{\beta\dot\beta i}) = 0$. Hence

$$\begin{aligned}(\Delta_{\alpha i}\Delta^{j}_{\dot\alpha} + \Delta^{j}_{\dot\alpha}\Delta_{\alpha i})\kappa_A &= -\Phi^{\gamma\dot\gamma j}_{\alpha\dot\alpha i}\Delta_{\gamma\dot\gamma}\kappa_A + \Delta_{\alpha\dot\alpha}\Phi\kappa_A - \Delta_{\alpha\dot\alpha}\Phi\kappa_A \\ &= -\Phi^{\gamma\dot\gamma j}_{\alpha\dot\alpha i}\Delta_{\gamma\dot\gamma}\kappa_A\,.\end{aligned}$$

For a connection $(A_{\alpha\dot\alpha}, \omega_{\alpha i}, \omega^{j}_{\dot\alpha})$ define

$$D_{\alpha\dot\alpha} \equiv \Delta_{\alpha\dot\alpha} + A_{\alpha\dot\alpha}\,,$$
$$Q_{\alpha i} \equiv \Delta_{\alpha i} + \omega_{\alpha i}\,,$$
$$Q^{j}_{\dot\alpha} \equiv \Delta^{j}_{\dot\alpha} + \omega^{j}_{\dot\alpha}\,.$$

For v^a, a section of our vector bundle, we have

$$\begin{aligned} Fv^a &= (d + (A_{\alpha\dot\alpha}, \omega_{\alpha i}, \omega^{j}_{\dot\alpha}))\,(d + (A_{\beta\dot\beta}, \omega_{\beta k}, \omega^{l}_{\dot\beta}))v^a \\ &= ((D_{\alpha\dot\alpha}, Q_{\alpha i}, Q^{j}_{\dot\alpha}) + \Phi^{\beta\dot\beta j}_{\alpha\dot\alpha i})\,(D_{\beta\dot\beta}v^a + Q_{\beta i}v^a + Q^{j}_{\dot\beta}v^a) \\ &= [D_{\alpha\dot\alpha}, D_{\beta\dot\beta}]v^a + [D_{\alpha\dot\alpha}, Q_{\beta i}]v^a + [D_{\alpha\dot\alpha}, Q^{j}_{\dot\beta}]v^a \\ &\quad + [Q_{\alpha i}, Q_{\beta j}]v^a + [Q^{i}_{\dot\alpha}, Q^{j}_{\dot\beta}]v^a + [Q_{\alpha i}, Q^{j}_{\dot\beta}]v^a + \Phi^{\beta\dot\beta j}_{\alpha\dot\alpha i} D_{\beta\dot\beta}v^a\,. \end{aligned}$$

If the connection is integrable along super light rays, we obtain

$$[Q_{\alpha i}, Q_{\beta j}] = W_{ij}\epsilon_{\alpha\beta}\,, \qquad [Q^{i}_{\dot\alpha}, Q^{j}_{\dot\beta}] = W^{ij}\epsilon_{\dot\alpha\dot\beta}\,,$$
$$[D_{\alpha\dot\alpha}, Q_{\beta j}] = \chi_{\dot\alpha j}\epsilon_{\alpha\beta}\,, \qquad [D_{\alpha\dot\alpha}, Q^{j}_{\beta}] = \chi^{j}_{\alpha}\epsilon_{\alpha\beta}\,,$$
$$[Q_{\alpha i}, Q^{j}_{\beta}] = -\Phi^{\beta\dot\beta j}_{\alpha\dot\alpha i} D_{\beta\dot\beta}\,.$$

Note that the last equation is true, at first, only for sections of E and not for sections of $E \otimes \Omega^\bullet M$ but by the previous calculation it can be extended to $E \otimes \Omega^\bullet M$: By the above

$$q_{\alpha j}(\omega^{k}_{\dot\alpha}) + q^{k}_{\dot\alpha}(\omega_{\alpha j}) + [\omega_{\alpha j}, \omega^{k}_{\dot\alpha}] = -\Phi^{\gamma\dot\gamma k}_{\alpha\dot\alpha j} A_{\gamma\dot\gamma}$$

and thus

$$[\Delta_{\alpha j} + \omega_{\alpha j}, \Delta^k_{\dot\alpha} + \omega^k_{\dot\alpha}] = -\Phi^{\gamma\dot\gamma k}_{\alpha\dot\alpha j} D_{\gamma\dot\gamma} - \omega^k_{\dot\alpha}\Delta_{\alpha j} - \omega_{\alpha j}\Delta^k_{\dot\alpha} + \omega_{\alpha j}\Delta^k_{\dot\alpha} + \omega^k_{\dot\alpha}\Delta_{\alpha j}$$
$$= -\Phi^{\gamma\dot\gamma k}_{\alpha\dot\alpha j} D_{\gamma\dot\gamma}$$

on all of $E \otimes \Omega^\bullet M$. Here we assume that the Frobenius form corresponds to convolution and $D_{\alpha\dot\alpha}, Q_{\alpha i}, Q^j_{\dot\alpha}$ are written with respect to such a basis. The above equation is just

$$[Q_{\alpha i}, Q^j_{\dot\alpha}] = -\delta^j_i D_{\alpha\dot\alpha}\,.$$

Using the Bianchi identities one may define λ_α and $\lambda_{\dot\alpha}$ by

$$Q_{\alpha i} W_{jk} = \epsilon_{ijk}\lambda_\alpha\,, \qquad Q^i_{\dot\alpha} W^{jk} = \epsilon_{ijk}\lambda_{\dot\alpha}\,.$$

5.3. THE EULER OPERATOR

We define, only locally, the Euler operator by

$$\mathcal{D} = \theta^{\alpha i}\partial/\partial\theta^{\alpha i} + \theta^{\dot\alpha}_i \partial/\partial\theta^{\dot\alpha}_i\,.$$

Recall that

$$Q_{\alpha i} = g^{\beta j}_{\alpha i}\Delta_{\beta j} + \omega_{\alpha i}\,, \qquad Q^i_{\dot\alpha} = g^{i\dot\beta}_{\dot\alpha j}\Delta^j_{\dot\beta} + \omega^i_{\dot\alpha}\,.$$

To describe $\mathcal{D}$ in terms of $Q_{\alpha i}$ and $Q^i_{\dot\alpha}$ we shall need the following:

Lemma 5.2. *There are coordinates $x^a, \theta'^{\alpha i}, \theta'^{\dot\alpha}_i$ such that*

$$g^{\beta j}_{\alpha i} = I^{\beta j}_{\alpha i} \bmod (\mathrm{Nil})^2\,, \qquad g^{i\dot\beta}_{\dot\alpha j} = I^{i\dot\beta}_{\dot\alpha j} \bmod (\mathrm{Nil})^2\,.$$

Proof. First form new functions

$$\tilde g^{\beta j}_{\alpha i} \equiv g^{\beta j}_{\alpha i}(x_l, \theta_l, 0)\,, \qquad \tilde g^{i\dot\beta}_{\dot\alpha j} \equiv g^{i\dot\beta}_{\dot\alpha j}(x_r, \theta_r, 0)\,.$$

More specifically, since $x^a_l = x^a + \mathrm{i}H^a$,

$$\tilde g^{\beta j}_{\alpha i}(x, \theta_l, \theta_r) = g^{\beta j}_{\alpha i}(x + \mathrm{i}H, \theta_l, 0)$$
$$= g(x^a, \theta_l, 0) + \mathrm{i}\frac{\partial g}{\partial x^a}(x^a, \theta_l, 0)H^a - \frac{1}{2}\frac{\partial^2 g}{\partial x^a \partial x^b}H^a H^b + \cdots\,.$$

The above sum is finite since H^a is nilpotent. Define $\tilde g^{i\dot\beta}_{\dot\alpha j}$ similarly. Clearly

$$q^k_{\dot\gamma}\tilde g^{\beta j}_{\alpha i} = 0\,, \qquad q_{\gamma k}\tilde g^{i\dot\beta}_{\dot\alpha j} = 0\,.$$

Now note that

$$\theta'^{\beta j} \equiv \tilde g^{\beta j}_{\alpha i}\theta^{\alpha i}\,, \qquad \theta'^{\dot\beta}_j \equiv \tilde g^{i\dot\beta}_{\dot\alpha j}\theta^{\dot\alpha}_i$$

are well defined odd coordinates such that $d\theta'^{\beta j}$ and $d\theta'^{\dot\beta}_j$ span $\Omega^1_l M$ and $\Omega^1_r M$ respectively. (Recall that $\Omega^1_{l,r} M$ are defined as quotient bundles.)

Since

$$d\theta'^{\beta j} = d(\tilde{g}^{\beta j}_{\alpha i})\theta^{\alpha i} + \tilde{g}^{\beta j}_{\alpha i} d\theta^{\alpha i} = g^{\beta j}_{\alpha i} d\theta^{\alpha i} \bmod (\mathrm{Nil})^2 \cdot \Omega^1 M + \Omega^1_0 M,$$

we have

$$d\theta'^{\beta j} = s^{\beta}_{+} \otimes e^j \bmod (\mathrm{Nil})^2 \cdot \Omega^1 M .$$

If $\hat{g}$ is the isomorphism from $\Omega^1_l M$ with basis $d\theta'^{\beta j}$ to $S^*_+ \otimes E^*$ with basis $s^{\alpha}_+ \otimes e^i$ then it is clear that

$$\hat{g}^{\beta j}_{\alpha i} = I^{\beta j}_{\alpha i} \bmod (\mathrm{Nil})^2 .$$

Similarly

$$\hat{g}^{i\dot{\beta}}_{\dot{\alpha}j} = I^{i\dot{\beta}}_{\dot{\alpha}j} \bmod (\mathrm{Nil})^2 . \qquad \square$$

Using the coordinates $\theta'^{\alpha i}$ and $\theta'^{\dot{\alpha}}_i$ from the lemma and dropping the use of the primes, we can now write the Euler operator as

$$\mathcal{D} = \theta^{\alpha i}\Delta_{\alpha i} + \theta^{\dot{\alpha}}_i \Delta^i_{\dot{\alpha}} + U^{\alpha\dot{\alpha}}\Delta_{\alpha\dot{\alpha}} + V^{\alpha i}\Delta_{\alpha i} + V^{\dot{\alpha}}_i \Delta^i_{\dot{\alpha}} + \theta^{\alpha i}\Gamma_{\alpha i} + \theta^{\dot{\alpha}}_i \Gamma^i_{\dot{\alpha}},$$

where $U^{\alpha\dot{\alpha}} \in (\mathrm{Nil})^2$ and $V^{\alpha i}, V^{\dot{\alpha}}_i \in (\mathrm{Nil})^3$. The $\Gamma_{\alpha i}$ and $\Gamma^i_{\dot{\alpha}}$ are $-$(the "Christoffel symbols" of $\Delta_{\alpha i}$ and $\Delta^i_{\dot{\alpha}}$). Also define

$$\hat{\mathcal{D}} = \theta^{\alpha i}\Delta_{\alpha i} + \theta^{\dot{\alpha}}_i \Delta^i_{\dot{\alpha}} .$$

Note that if we impose on a connection the transverse gauge condition

$$\theta^{\alpha i}\omega_{\alpha i} + \theta^{\dot{\alpha}}_i \omega^i_{\dot{\alpha}} = 0,$$

then

$$\hat{D} = \theta^{\alpha i} Q_{\alpha i} + \theta^{\dot{\alpha}}_i Q^i_{\dot{\alpha}} .$$

Also note that $\hat{D} = D + T$, where T is an operator which strictly increases nilpotency and is, of course, independent of any particular connection.

5.4. EQUIVALENCE OF DATA

We wish to show the equivalence of the following three types of data (see Harnad et al. [7] or Schnider and Wells [19]). We will be working thoughout this section over a neighborhood of M for which we have a choice of supercoordinates and a trivialization of our vector bundle.

i) Integrability along super light rays. The superconnection $(A_{\alpha\dot{\alpha}}, \omega_{\alpha i}, \omega^j_{\dot{\alpha}})$ subject to the constraints:

$$[Q_{\alpha i}, Q_{\beta j}] + [Q_{\beta i}, Q_{\alpha i}] = 0,$$

$$[Q^i_{\dot{\alpha}}, Q^j_{\dot{\beta}}] + [Q^j_{\dot{\alpha}}, Q^i_{\dot{\beta}}] = 0,$$

$$[Q_{\alpha i}, Q^j_{\dot{\beta}}] = -\delta^j_i D_{\alpha\dot{\alpha}},$$

and the following "transverse" gauge condition:

$$\theta^{\alpha i}\omega_{\dot{\alpha} i} + \theta^{i}_{\dot{\alpha}}\omega^{i}_{\dot{\alpha}} = 0 .$$

Note that the first two constraints are equivalent to

$$[Q_{\alpha i}, Q_{\beta j}] = \epsilon_{\alpha\beta} W_{ij} , \qquad [Q^{i}_{\dot{\alpha}}, Q^{j}_{\dot{\beta}}] = \epsilon_{\dot{\alpha}\dot{\beta}} W^{ij}$$

for some superfields W_{ij} and W^{ij}. We have thus already shown that a connection with curvature vanishing along super light rays satisfies these constraints. Likewise the constraints imply, via the Bianchi identity, that the curvature F has the form written before for integrability along super light rays.

We also note that the "transverse" gauage condition may always be validly applied, i.e. given a connection, we may always find a second connection gauge equivalent to it which satisfies this condition.

ii) The superfield equations. The superfields $\{A_{\alpha\dot{\alpha}}, \lambda_{\alpha}, \lambda_{\dot{\alpha}}, \chi^{i}_{\alpha}, \chi_{i\dot{\alpha}}, W_{i}, W^{i}\}$ (where $W_i \equiv \epsilon_{ijk} W^{jk}$ and $W^{i} \equiv \epsilon^{ijk} W_{jk}$), subject to the superfield equations written below, with the rd dropped. In addition, there is a certain set of relations, called the $\hat{\mathcal{D}}$-recursions, which are defined in terms of $\hat{\mathcal{D}}$. The $\hat{\mathcal{D}}$-recursions:

$$\hat{\mathcal{D}} W_{jk} = \epsilon_{ijk}\theta^{i\alpha}\lambda_{\alpha} + \theta^{\dot{\alpha}}_{j}\chi_{k\dot{\alpha}} - \theta^{\dot{\alpha}}_{k}\chi_{j\dot{\alpha}} ,$$

$$\hat{\mathcal{D}} W^{jk} = \epsilon^{ijk}\theta^{\dot{\alpha}}_{i}\lambda_{\dot{\alpha}} + \theta^{j\alpha}\chi^{k}_{\alpha} - \theta^{k\alpha}\chi^{j}_{\alpha} ,$$

$$\hat{\mathcal{D}} A_{\alpha\dot{\beta}} = -\epsilon_{\alpha\beta}\theta^{i\beta}\chi_{i\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\theta^{\dot{\alpha}}_{i}\chi^{i}_{\alpha} ,$$

$$\hat{\mathcal{D}}\chi_{i\dot{\alpha}} = 2\theta^{j\beta} D_{\beta\dot{\alpha}} W_{ji} + 2\theta^{\dot{\beta}}_{i} f_{\dot{\alpha}\dot{\beta}} + 2\theta^{\dot{\beta}}_{j}\epsilon_{\dot{\alpha}\dot{\beta}}[W^{jk}, W_{ik}] - \tfrac{1}{2}\theta^{\dot{\beta}}_{i}\epsilon_{\dot{\alpha}\dot{\beta}}[W^{kl}, W_{kl}] ,$$

$$\hat{\mathcal{D}}\chi^{i}_{\alpha} = 2\theta^{\dot{\beta}}_{j} D_{\alpha\dot{\beta}} W^{ji} + 2\theta^{i\beta} f_{\alpha\beta} + 2\theta^{j\beta}\epsilon_{\alpha\beta}[W_{jk}, W^{ik}] - \tfrac{1}{2}\theta^{i\beta}\epsilon_{\alpha\beta}[W_{kl}, W^{kl}] ,$$

$$\hat{\mathcal{D}}\lambda_{\alpha} = \tfrac{1}{2}\theta^{i\beta}\epsilon_{\beta\alpha}[W_{ij}, W_{kl}]\epsilon^{jkl} + \theta^{\dot{\beta}}_{i}\epsilon^{ijk} D_{\alpha\dot{\beta}} W_{jk} ,$$

$$\hat{\mathcal{D}}\lambda_{\dot{\alpha}} = \tfrac{1}{2}\theta^{\dot{\beta}}_{i}\epsilon_{\dot{\alpha}\dot{\beta}}[W^{ij}, W^{kl}]\epsilon_{jkl} + \theta^{i\alpha}\epsilon_{ijk} D_{\alpha\dot{\alpha}} W^{jk} ,$$

$$\hat{\mathcal{D}} f_{\alpha\beta} = \tfrac{1}{2}\theta^{i\gamma}\epsilon^{\dot{\alpha}\dot{\beta}}[\epsilon_{\beta\gamma} D_{\alpha\dot{\alpha}}\chi_{i\dot{\beta}} + \epsilon_{\alpha\gamma} D_{\beta\dot{\alpha}}\chi_{i\dot{\beta}}] + \theta^{\dot{\gamma}}_{i}[D_{\alpha\dot{\gamma}}\chi^{i}_{\beta} + D_{\beta\dot{\gamma}}\chi^{i}_{\alpha}] ,$$

$$\hat{\mathcal{D}} f_{\dot{\alpha}\dot{\beta}} = \tfrac{1}{2}\theta^{\dot{\gamma}}_{i}\epsilon^{\alpha\beta}[\epsilon_{\dot{\beta}\dot{\gamma}} D_{\alpha\dot{\alpha}}\chi^{i}_{\beta} + \epsilon_{\dot{\alpha}\dot{\gamma}} D_{\alpha\dot{\beta}}\chi^{i}_{\beta}] + \tfrac{1}{2}\theta^{i\gamma}[D_{\gamma\dot{\alpha}}\chi_{i\dot{\beta}} + D_{\gamma\dot{\beta}}\chi_{i\dot{\alpha}}] .$$

iii) The (reduced) field equations. The component fields $\{A_{rd\,\alpha\dot{\alpha}}, \lambda_{rd\,\alpha}, \lambda_{rd\,\dot{\alpha}}, \chi^{i}_{rd\,\alpha}, \chi_{rd\,i\dot{\alpha}}, W_{rd\,i}, W^{i}_{rd}\}$ subject to the (reduced) field equations:

$$\epsilon^{\alpha\beta} D_{rd\,\alpha\dot{\beta}}\lambda_{rd\,\beta} + [\chi_{rd\,i\dot{\beta}}, W^{i}_{rd}] = 0 ,$$

$$\epsilon^{\dot{\alpha}\dot{\beta}} D_{rd\,\alpha\dot{\alpha}}\lambda_{rd\,\dot{\beta}} + [\chi^{i}_{rd\,\alpha}, W_{rd\,i}] = 0 ,$$

$$\epsilon^{\alpha\beta} D_{rd\,\alpha\dot{\beta}}\chi^{j}_{rd\,\beta} + [\chi_{rd\,i\dot{\beta}}, W_{rd\,k}]\epsilon^{ijk} - [\lambda_{rd\,\dot{\beta}}, W^{j}_{rd}] = 0 ,$$

$$\epsilon^{\dot{\alpha}\dot{\beta}} D_{rd\,\alpha\dot{\alpha}}\chi_{rd\,j\dot{\beta}} + [\chi^{i}_{rd\,\alpha}, W^{k}_{rd}]\epsilon_{ijk} - [\lambda_{rd\,\alpha}, W_{rd\,j}] = 0 ,$$

$$\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}D_{rd\,\alpha\dot\alpha}D_{rd\,\beta\dot\beta}W_{rd\,j} + 2\{[[W^i_{rd}, W_{rd\,j}], W_{rd\,i}] - [[W^i_{rd}, W_{rd\,i}], W_{rd\,j}]\}$$
$$+ \epsilon^{\dot\alpha\dot\beta}\{\chi_{rd\,j\dot\alpha}, \lambda_{rd\,\dot\beta}\} - \tfrac{1}{2}\epsilon_{ijk}\epsilon^{\alpha\beta}\{\chi^i_{rd\,\alpha}, \chi^k_{rd\,\beta}\} = 0,$$

$$\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}D_{rd\,\alpha\dot\alpha}D_{rd\,\beta\dot\beta}W^j_{rd} + 2\{[[W_{rd\,i}, W^j_{rd}], W^i_{rd}] - [[W_{rd\,i}, W^i_{rd}], W^j_{rd}]\}$$
$$+ \epsilon^{\alpha\beta}\{\chi^j_{rd\,\alpha}, \lambda_{rd\,\beta}\} - \tfrac{1}{2}\epsilon^{ijk}\epsilon^{\dot\alpha\dot\beta}\{\chi_{rd\,i\dot\alpha}, \chi_{rd\,k\dot\beta}\} = 0,$$

$$\epsilon^{\alpha\beta}D_{rd\,\alpha\dot\beta}f_{rd\,\gamma\beta} + \epsilon^{\dot\alpha\dot\gamma}D_{rd\,\gamma\dot\alpha}f_{rd\,\dot\gamma\dot\beta} + \{\chi^k_{rd\,\gamma}, \chi_{rd\,k\dot\beta}\} + \{\lambda_{rd\,\gamma}, \lambda_{rd\,\dot\beta}\}$$
$$+ [W^i_{rd}, D_{rd\,\gamma\dot\beta}W_{rd\,i}] + [W_{rd\,i}, D_{rd\,\gamma\dot\beta}W^i_{rd}] = 0.$$

Proof. Obviously, ii) $\Rightarrow$ iii) is just trivially applying reduction. The proof of i) $\Rightarrow$ ii) follows through just as it is done in Harnad et al. [7]. We repeat their argument here.

We first have the superfield curvature tensors $f_{\alpha\beta}$ and $f_{\dot\alpha\dot\beta}$ defined by

$$[D_{\alpha\dot\alpha}, D_{\beta\dot\beta}] = \epsilon_{\dot\alpha\dot\beta}f_{\alpha\beta} + \epsilon_{\alpha\beta}f_{\dot\alpha\dot\beta}\,.$$

Using the constraint equations and the Bianchi identity, we obtain superfields, $\lambda_\alpha, \lambda_{\dot\alpha}, \chi^i_\beta, \chi_{\dot\beta}$ satisfying

$$Q_{\alpha i}W_{jk} = \epsilon_{ijk}\lambda_\alpha\,, \tag{1}$$
$$Q^i_{\dot\alpha}W^{jk} = \epsilon^{ijk}\lambda_{\dot\alpha}\,, \tag{2}$$
$$[Q_{\alpha i}, D_{\beta\dot\gamma}] = \epsilon_{\alpha\beta}\chi_{i\dot\gamma}\,, \tag{3}$$
$$[Q^i_{\dot\beta}, D_{\alpha\dot\gamma}] = \epsilon_{\dot\beta\dot\gamma}\chi^i_\alpha\,, \tag{4}$$

and also the equations

$$Q^i_{\dot\alpha}W_{jk} = \delta^i_j\chi_{k\dot\alpha} - \delta^i_k\chi_{j\dot\alpha}\,, \qquad Q_{\alpha k}W^{ij} = \delta^i_k\chi^j_\alpha - \delta^j_k\chi^i_\alpha\,, \tag{5}$$
$$Q_{\alpha i}\chi_{j\dot\alpha} = 2D_{\alpha\dot\alpha}W_{ij}\,, \qquad Q^i_{\dot\alpha}\chi^j_\alpha = 2D_{\alpha\dot\alpha}W^{ij}\,, \tag{6}$$
$$Q^j_{\dot\beta}\chi_{i\dot\alpha} = 2\delta^j_i f_{\dot\alpha\dot\beta} + 2\epsilon_{\dot\alpha\dot\beta}[W^{jk}, W_{ik}] - \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\delta^j_i[W^{kn}, W_{kn}]\,, \tag{7}$$
$$Q_{\beta i}\chi^j_\alpha = 2\delta^j_i f_{\alpha\beta} + 2\epsilon_{\alpha\beta}[W_{ki}, W^{kj}] - \tfrac{1}{2}\epsilon_{\alpha\beta}\delta^j_i[W_{kn}, W^{kn}]\,, \tag{8}$$
$$Q_{\gamma i}f_{\alpha\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}[\epsilon_{\beta\gamma}D_{\alpha\dot\alpha}\chi_{i\dot\beta} + \epsilon_{\alpha\gamma}D_{\beta\dot\alpha}\chi_{i\dot\beta}]\,, \tag{9}$$
$$Q_{\gamma i}f_{\dot\alpha\dot\beta} = \tfrac{1}{2}[D_{\gamma\dot\alpha}\chi_{i\dot\beta} + D_{\gamma\dot\beta}\chi_{i\dot\alpha}]\,, \tag{10}$$
$$Q^i_{\dot\gamma}f_{\dot\alpha\dot\beta} = \tfrac{1}{2}\epsilon^{\alpha\beta}[\epsilon_{\dot\beta\dot\gamma}D_{\alpha\dot\alpha}\chi^i_\beta + \epsilon_{\dot\alpha\dot\gamma}D_{\alpha\dot\beta}\chi^i_\beta]\,, \tag{11}$$
$$Q^i_{\dot\gamma}f_{\alpha\beta} = \tfrac{1}{2}[D_{\alpha\dot\gamma}\chi^i_\beta + D_{\beta\dot\gamma}\chi^i_\alpha]\,. \tag{12}$$

Applying $Q_{\alpha i}$ and $Q^j_{\dot\alpha}$ to eqs. (1) and (2) gives

$$Q_{\alpha i}\lambda_{\dot\beta} = \epsilon_{ijk}D_{\alpha\dot\beta}W^{jk}\,, \tag{13}$$
$$Q_{\alpha i}\lambda_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}[W_{ij}, W_{kl}]\epsilon^{jkl}\,, \tag{14}$$

$$Q^i_{\dot\alpha}\lambda_{\dot\beta} = \tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}[W^{ij}, W^{kl}]\epsilon_{jkl}, \tag{15}$$

$$Q^i_{\dot\alpha}\lambda_\beta = \epsilon^{ijk}D_{\beta\dot\alpha}W_{jk}. \tag{16}$$

Applying

$$D_{\alpha\dot\alpha} = \tfrac{1}{6}(Q_{\alpha i}Q^i_{\dot\alpha} + Q^i_{\dot\alpha}Q_{\alpha i})$$

to $\lambda_\beta, \lambda_{\dot\beta}, \chi^j_\beta, \chi_{j\dot\beta}$ and using eqs. (1)–(16) gives the first four superfield equations. Apply $Q_{\beta j}$ to the second superfield equation, $Q^j_{\dot\alpha}$ to the first superfield equation, and $Q_{\gamma j}$ to the third superfield equation, to give respectively the last three superfield equations.

Apply $\hat{\mathcal{D}} = \theta^{\alpha i}Q_{\alpha i} + \theta^{\dot\alpha}_i Q^i_{\dot\alpha}$ to $W_{ij}, W^{kl}, \chi_{i\dot\beta}, \chi^j_\beta, \lambda_\alpha, \lambda_{\dot\alpha}, f_{\alpha\beta}, f_{\dot\alpha\dot\beta}$ and use eqs. (1)–(16) to yield the $\hat{\mathcal{D}}$-recursions. We note that we have

$$[\Delta_{\beta i}, \Delta_{\alpha\dot\alpha}] = [\Delta^i_{\dot\beta}, \Delta_{\alpha\dot\alpha}] = 0.$$

(This follows from $\Delta^2 = -\Delta\Phi - \Phi\Delta$ or from a local calculation where the "Christoffel symbols" of $\Delta_{\beta i}$ and $\Delta^i_{\dot\beta}$ respectively cause cancellation of $[q_{\beta i}, \partial_{\alpha\dot\alpha}]$ and $[q^i_{\dot\beta}, \partial_{\alpha\dot\alpha}]$.) Thus

$$[\hat{\mathcal{D}}, D_\alpha] = \hat{\mathcal{D}}A_{\alpha\dot\alpha}.$$

This then gives

$$\hat{\mathcal{D}}A_{\alpha\dot\alpha} = \theta^{\beta i}\chi_{i\dot\alpha}\epsilon_{\beta\alpha} + \theta^{\dot\beta}_i\chi^i_\alpha\epsilon_{\dot\beta\dot\alpha}.$$

Applying $\hat{\mathcal{D}}$ to $Q_{\alpha i} = \Delta_{\alpha i} + \omega_{\alpha i}$ and $Q^i_{\dot\alpha} = \Delta^i_{\dot\alpha} + \omega^i_{\dot\alpha}$ gives us

$$(1+\hat{\mathcal{D}})\omega_{\alpha i} = 2\epsilon_{\alpha\beta}\theta^{\beta j}W_{ij} + 2\theta^{\dot\alpha}_i A_{\alpha\dot\alpha},$$

$$(1+\hat{\mathcal{D}})\omega^i_{\dot\alpha} = 2\epsilon_{\dot\alpha\dot\beta}\theta^{\dot\beta}_j W^{ij} + 2\theta^{\alpha i}A_{\alpha\dot\alpha}.$$

In proving iii) $\Rightarrow$ ii) we must first take the $\hat{D}$-recursions as defining $A_{\alpha\dot\alpha}, \lambda_\alpha, \lambda_{\dot\alpha}, \chi^i_\alpha, \chi_{i\dot\alpha}, W_i, W^i$ inductively on their nilpotency. We note that this is possible since $\hat{D} = D + T$, where T strictly increases nilpotency and is independent of the connection. Next we are trying to show that

$$G = 0,$$

given that $G_{rd} = 0$ where G is the left-hand side of one of the superfield equations. It is actually a system of equations

$$\overset{k}{G} = 0,$$

where $\overset{k}{G} \in (\mathrm{Nil})^k$. Assume $\overset{0}{G} = \overset{1}{G} = \cdots = \overset{n-1}{G} = 0$ where $\overset{0}{G} \equiv G_{rd}$. Now

$$\mathcal{D}\overset{n}{G} = n\overset{n}{G} = \overbrace{(\mathcal{D}G)}^{n}.$$

Also

$$\begin{aligned}\overbrace{(\mathcal{D}G)}^{n} = {} & \theta^{\alpha i} Q_{\alpha i} \overset{\leqslant n}{G} + \theta_i^{\dot\alpha} Q^i_{\dot\alpha} \overset{\leqslant n}{G} + U^{\alpha\dot\alpha}\Delta_{\alpha\dot\alpha}\overset{<n}{G} + V^{\alpha i}\Delta_{\alpha i}\overset{<n}{G} \\ & + V_i^{\dot\alpha}\Delta^i_{\dot\alpha}\overset{<n}{G} + \theta^{\alpha i}\Gamma_{\alpha i}\overset{<n}{G} + \theta_i^{\dot\alpha}\Gamma^i_{\dot\alpha}\overset{<n}{G},\end{aligned}$$

where $\overset{<n}{G}$ is $\overset{l}{G}$ for some $l < n$, i.e. zero, and $\overset{\leqslant n}{G}$ is just $\overset{n}{G}$. Thus

$$\overbrace{(\mathcal{D}G)}^{n} = \overbrace{(\theta^{\alpha i} Q_{\alpha i} G + \theta_i^{\dot\alpha} Q^i_{\dot\alpha} G)}^{n}.$$

The $\mathcal{D}$-recursions of Harnad et al. [2] are valid as $\hat{\mathcal{D}}$-recursions by just replacing $\mathcal{D}$ everywhere with $\hat{\mathcal{D}}$. We can use the $\hat{\mathcal{D}}$-recursions in exactly the same manner as Harnad et al. [7] use the $\mathcal{D}$-recursions, to show recursively that if G is the left-hand side of one of the $N = 3$ SSYM field equations then

$$n\overset{n}{G} = \overbrace{(\mathcal{D}G)}^{n} = \overbrace{(\hat{\mathcal{D}}G)}^{n} = 0.$$

This completes iii) $\Rightarrow$ ii).

Now turn to the proof of ii) $\Rightarrow$ i). Similarly as in Harnad et al. [7] we have, assuming i) (integrability along super light rays):

$$\begin{aligned}(1 + \hat{\mathcal{D}})\omega_{\alpha i} &= 2\epsilon_{\alpha\beta}\theta^{\beta j} W_{ij} + 2\theta_i^{\dot\alpha} A_{\alpha\dot\alpha}, \\ (1 + \hat{\mathcal{D}})\omega^i_{\dot\alpha} &= 2\epsilon_{\dot\alpha\dot\beta} W_{ij} + 2\theta^{\alpha i} A_{\alpha\dot\alpha}.\end{aligned}$$

One can thus use this to define recursively

$$\overbrace{\{(1 + \mathcal{D})\omega_{\alpha i}\}}^{n} = \overbrace{2\epsilon_{\alpha\beta}\theta^{\beta j} W_{ij} + 2\theta_i^{\dot\alpha} A_{\alpha\dot\alpha}}^{n} + \overbrace{T\omega_{\alpha i}}^{n},$$

where $T = \hat{\mathcal{D}} - \mathcal{D}$. Note that $T^{\beta j}_{\alpha i}$ as an operator strictly increases the nilpotency since

$$T = U^{\beta\dot\beta}\Delta_{\beta\dot\beta} + V^{\beta j}\Delta_{\beta j} + V_j^{\dot\beta}\Delta^j_{\dot\beta} + \theta^{\beta j}\Gamma_{\beta j} + \theta_j^{\dot\beta}\Gamma^j_{\dot\beta},$$

where $U^{\beta\dot\beta} \in (\mathrm{Nil})^2, V^{\beta j}, V_j^{\dot\beta} \in (\mathrm{Nil})^3$ and $\Gamma_{\beta j}, \Gamma^j_{\dot\beta}$ locally are just matrices or zero. Thus

$$\overbrace{T\omega_{\alpha i}}^{n} = \overbrace{T(\sum_{l<n} \overset{l}{\omega}_{\alpha i})}^{n}.$$

One can similarly define $\omega^i_{\dot\alpha}$ recursively.

We will want to prove eqs. (1)–(16), just as is done in Harnad et al. [7], which in turn imply the constraint equations for integrability of the connection along super light rays. As is done there, apply $(1 + \mathcal{D})$ to both sides of the equation we are trying to prove, $G = 0$, and use induction on the nilpotency.

We have $\overset{0}{G} = 0$ for eqs. (1)–(16), using the $\hat{\mathcal{D}}$-recursions. Assume $\overset{l}{G} = 0$ for $l < n$. Then

$$\overbrace{(1 + \mathcal{D})G}^{n} = \overbrace{(1 + \hat{\mathcal{D}} + T)G}^{n}$$

$$= \overbrace{(1 + \hat{\mathcal{D}})G}^{n} + \overbrace{T(\sum_{l<n} \overset{l}{G})}^{n} = \overbrace{(1 + \hat{\mathcal{D}})G}^{n}.$$

One can use the $\hat{\mathcal{D}}$-recursions in exactly the same way as Harnad et al. [7] use the $\mathcal{D}$-recursions to show that this last expression is zero for $G = 0$ being one of eqs. (1)–(16). To show that these equations imply the constraint equations we apply $2 + \hat{\mathcal{D}}$ and a recursive argument on the nilpotency to both sides of each of the constraint equations. We refer the reader to ref. [7, p. 619], where Harnad et al. show, as an example, that

$$\overbrace{(2 + \mathcal{D})(\{Q_{\alpha i}, Q_{\beta j}\} - 2\epsilon_{\alpha\beta} W_{ij})}^{n} = \overbrace{(2 + \hat{\mathcal{D}})(\{Q_{\alpha i}, Q_{\beta j}\} - 2\epsilon_{\alpha\beta} W_{ij})}^{n} = 0,$$

using eqs. (1)–(16). This completes the proof of ii) $\Rightarrow$ i) and thus completes our proof of the equivalence of the three sets of data. $\square$

We note that i) $\Leftrightarrow$ iii) tells us that the data of the reduced fields determines a unique superconnection (up to gauge equivalence). For if we had two superconnections corresponding to the same set of reduced fields we could then find for each a superconnection which is gauge equivalent and which satisfies the "transverse" gauge condition in a common fixed choice of super coordinates. These two connections would then have to be equal to each other by the equivalence of data proven above.

6. Vector bundles and SSYM fields

It is now a well established procedure to show the equivalence of $N = 3$ superconnections integrable along super light rays and vector bundles over the space of super light rays which vanish on normal quadrics. The reader may refer to Manin [15] or Schnider and Wells [19]. Recall the double fibration:

$$\begin{array}{ccc} & F & \\ \overset{\rho}{\swarrow} & & \overset{\pi}{\searrow} \\ \mathcal{N}^{5|6} & & M^{4|12}. \end{array}$$

We present here the argument of Manin [15, pp. 73–74], to construct from a connection on $M^{4|12}$ which is integrable along super light rays, a vector bundle on $\mathcal{N}^{5|6}$ which is trivial on normal quadrics.

Assume the fibres of ρ, i.e. the super light rays of M, are connected. Let $(\mathcal{E}_M, \nabla)$ be a vector bundle with connection on M, which is integrable along super light rays and which has zero monodromy along these fibres. Let $TF/\mathcal{N} = \ker(\rho_*)$ and let $\nabla_{F/\mathcal{N}}$ be the composition

$$\pi^*\mathcal{E}_M \overset{\pi^*\nabla}{\to} \pi^*\mathcal{E}_M \otimes \pi^*\Omega^1 M \overset{\mathrm{id}\otimes\mathrm{res}}{\to} \Omega^1 F/\mathcal{N},$$

where res is the restriction to $TF/\mathcal{N}$. Define $\mathcal{E}'_F \equiv \ker(\nabla_{F/\mathcal{N}})$. Since $\nabla_{F/\mathcal{N}}$ has no curvature or monodromy and the fibres of ρ are connected, we have that $\mathcal{E}_\mathcal{N} = \rho_*\mathcal{E}'_F$ is a locally free sheaf of $\mathcal{A}_\mathcal{N}$-modules on $\mathcal{N}$. Furthermore, this sheaf will be trivial when restricted to normal quadrics.

Now let $\mathcal{E}_\mathcal{N}$ be a vector bundle over $\mathcal{N}$ which is trivial over normal quadrics. Let $\mathcal{E}_F = \rho^*(\mathcal{E}_\mathcal{N})$. Since $\mathcal{E}_F$ is trivial on the fibres of π, we have $\mathcal{E}_F = A_F \otimes_{A_M} \mathcal{E}_0$ for some sheaf $\mathcal{E}_0$, which we can identify with some sheaf $\mathcal{E}_M$ on M. The vector bundle $\mathcal{E}_M$ will then, by its construction have zero monodromy along any null geodesic. A connection on $\mathcal{E}_M$ can be defined by a straightforward generalization of the Sparling–Ward splitting outlined by Shnider and Wells [19, pp. 52–53].

Let $\mathcal{N}^{5|6}$ be a space of super light rays constructed for a complex conformal spacetime M^4. Assume also that M^4 is civilized and reflexive and initially that M^4 is a Stein open set over which our vector bundle $\mathcal{E}_{rd}$ is trivial and which is a supercoordinate chart for its extension $M^{4|12}$. The above establishes the following theorem:

Theorem 6.1. *There is a one to one correspondence between equivalence classes of*

- Solutions to the N=3 SSYM equations on of a complex conformal spacetime M^4 with no monodromy on any null line l, and

- Super vector bundles over the space of super light rays $\mathcal{N}^{5|6}$, which are trivial over normal embedded $\mathbb{P}_1 \times \mathbb{P}_1$.

We may now piece together the local versions of this theorem to produce a global version in the manner à la LeBrun [12, p. 1059]. We first cover our spacetime with convex neighborhoods for which the theorem already holds. The theorem will also be true on their overlaps.

Over the image of each of these in the space of super light rays we obtain, via the correspondence, a super vector bundle. On an overlap we have uniqueness up to isomorphism and thus an automorphism of the super vector bundle over it. On the reduced level this automorphism is the identity. But the identity has only a unique extension over our overlap. Thus we may piece together uniquely the super vector bundles over the images to obtain a unique super vector bundle over the entire space of super light rays which is trivial over normal quadrics.

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